

EILENBERG–MACLANE MAPPING ALGEBRAS AND HIGHER DISTRIBUTIVITY UP TO HOMOTOPY

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ABSTRACT. Primary cohomology operations, i.e., elements of the Steenrod algebra, are given by homotopy classes of maps between Eilenberg–MacLane spectra. Such maps (before taking homotopy classes) form the topological version of the Steenrod algebra. Composition of such maps is strictly linear in one variable and linear up to coherent homotopy in the other variable. To describe this structure, we introduce a hierarchy of higher distributivity laws, and prove that the topological Steenrod algebra satisfies all of them. We show that the higher distributivity laws are homotopy invariant in a suitable sense. As an application of 2-distributivity, we construct a new derivation of degree -2 of the mod 2 Steenrod algebra.

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1. INTRODUCTION

The elements of the Steenrod algebra are primary cohomology operations, which are given by homotopy classes of maps between Eilenberg–MacLane spectra. Richer information is contained in the *mapping spaces* between Eilenberg–MacLane spectra, which form a topological version of the Steenrod algebra that encodes secondary and all higher order cohomology operations. One of the great successes of algebraic topology was the complete computation of the Steenrod algebra, and the study of its algebraic properties [30] [1]. In contrast, the algebraic nature of higher order cohomology operations remained lesser known. Only secondary operations were studied in detail, notably in [2], [22], [23], [24], [21], [17], and [5].

One of the difficulties with higher order cohomology operations is that they do not form an algebra. While composition of elements in the Steenrod algebra is bilinear, composition

Date: March 23, 2017.

2010 Mathematics Subject Classification. Primary: 55S20; Secondary: 55P20, 55S10, 18G55.

Key words and phrases. higher distributivity, distributivity up to homotopy, higher cohomology operation, Eilenberg–MacLane spectrum, Steenrod algebra, Kristensen derivation, homotopy invariant, A-infinity morphism, topological abelian group, mapping theory, mapping algebra.

in the topological Steenrod algebra is not bilinear, but left linear (strictly) and right linear up to coherent homotopy.

In this paper, we introduce the notion of n^{th} order distributivity (for $n \geq 0$), which is similar to the notion of n^{th} order associativity or n^{th} order commutativity. Stasheff described higher order associativity via A_∞ -spaces, based on associahedra [34] [26, §I.1.6, II.1.6]. Other polytopes have been used to describe homotopy coherent algebraic structure, such as permutohedra, which encode higher order commutativity [37], or permuto-associahedra, which mix higher order associativity and commutativity [20]. A different (less strict) notion of higher distributivity is studied in [14, §6] using distributahedra. The higher order distributivity that we consider turns out to be based on higher dimensional cubes.

Our main result can be roughly stated as follows.

Theorem 1.1. *The topological Steenrod algebra satisfies infinitely many higher order coherent distributivity laws up to homotopy.*

Organization. In Section 2, we introduce the notion of *weakly bilinear mapping theory* (Definition 2.4), which has an addition, a multiplication, left linearity, but might not be strictly right linear. The motivational example is the *Eilenberg–MacLane mapping theory* \mathcal{EM} , given by finite products of Eilenberg–MacLane spectra and mapping spaces between them (Proposition 2.7 and Definition 2.9).

In Section 3, we define higher distributivity via higher dimensional cubes (Definition 3.6) and reformulate it as an inductive construction (Lemma 3.16). One of our main results is that a weakly bilinear mapping theory is ∞ -distributive, in a canonical way (Theorem 3.8 and Proposition 4.8).

Section 4 studies additional properties that the distributivity data might satisfy. These are used in the proof of the main result.

In Section 5, we study applications of higher distributivity to the mod 2 Steenrod algebra \mathcal{A} . We recall how the Kristensen derivation $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ can be obtained from the 1-distributivity of \mathcal{EM} ; Kristensen’s original construction used cochain operations. Using the 2-distributivity of \mathcal{EM} , we construct a new derivation $\lambda: \mathcal{A} \rightarrow \mathcal{A}$ of degree -2 (Proposition 5.16). We leave for future research an explicit algebraic formula for this derivation.

In Section 6, we show that n -distributivity is homotopy invariant in some appropriate sense. More precisely, our notion of higher distributivity has some strict structure built in: a strict addition, a strictly associative multiplication, and left linearity holding strictly. Corollary 6.10 says that n -distributivity is invariant with respect to Dwyer–Kan equivalences that preserve addition and multiplication strictly.

Appendices A and B collect for convenience some facts about Eilenberg–MacLane spectra.

Acknowledgments. The second author thanks the Max-Planck-Institut für Mathematik Bonn for its generous hospitality, as well as Tobias Barthel, David Blanc, Yaël Frégier, Lennart Meier, Fernando Muro, Irakli Patchkoria, Stefan Schwede, and Marc Stephan for useful conversations.

2. EILENBERG–MACLANE MAPPING ALGEBRAS

2.1. Mapping theories. Let \mathbf{Top} denote a convenient category of topological spaces, for instance compactly generated weakly Hausdorff spaces, so that internal hom objects Y^X exist for all objects X and Y of \mathbf{Top} . Let \mathbf{Top}_* denote the category of pointed spaces, with basepoints generically denoted by $0 \in X$.

Enrichment in \mathbf{Top}_* will always mean with respect to the smash product $X \wedge Y$ as symmetric monoidal structure. In a \mathbf{Top}_* -enriched category \mathcal{C} , we denote the composition map by

$$\mu: \mathcal{C}(B, C) \wedge \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$$

and write $\mu(x, y) = xy$ for short. Equivalently, this can be described by the map

$$\mu: \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$$

satisfying $\mu(0, y) = 0$ and $\mu(x, 0) = 0$ for all x and y .

Notation 2.1. We write $x \in \mathcal{C}$ if $x \in \mathcal{C}(A, B)$ for some objects A and B of \mathcal{C} . From now on, whenever an expression such as xy appears, it is understood that x and y must be composable, i.e., the target of y is the source of x .

Definition 2.2. (1) A **mapping theory** is a \mathbf{Top}_* -enriched small category \mathcal{T} with finite products $A \times B$ for all objects A, B in \mathcal{T} (and a terminal object $*$).
 (2) A **model** of the mapping theory \mathcal{T} is a \mathbf{Top}_* -enriched functor $F: \mathcal{T} \rightarrow \mathbf{Top}_*$ which preserves finite products (strictly).
 (3) A **mapping algebra** (\mathcal{T}, F) consists of a mapping theory \mathcal{T} together with a model F of \mathcal{T} .

Slight variations of this notion of *mapping algebra* appear in [6, §8], [3, §1], and [11].

Remark 2.3. The data of a mapping algebra (\mathcal{T}, F) can be encoded into a \mathbf{Top}_* -enriched category $\mathcal{T}\{F\}$, whose objects are those of \mathcal{T} plus a **distinguished object** \star , and mapping spaces are given by

$$\mathcal{T}\{F\}(A, B) = \begin{cases} \mathcal{T}(A, B) & \text{if } A, B \in \text{Ob } \mathcal{T} \\ F(B) & \text{if } A = \star, B \in \text{Ob } \mathcal{T} \\ * & \text{if } A \in \text{Ob } \mathcal{T}, B = \star \\ \{0, 1_\star\} & \text{if } A = B = \star. \end{cases}$$

Since F preserves finite products, the product $A \times B$ in \mathcal{T} is still a product in $\mathcal{T}\{F\}$. However, $\mathcal{T}\{F\}$ does not have products involving the distinguished object \star .

Using this construction, statements about mapping theories will have analogues about mapping algebras. The arguments apply as long as we never map *into* the distinguished object \star .

In our main application, \mathcal{T} will be the full subcategory of a \mathbf{Top}_* -enriched category \mathcal{C} . For any object X of \mathcal{C} , the functor $F_X := \mathcal{C}(X, -): \mathcal{T} \rightarrow \mathbf{Top}_*$ is a model of \mathcal{T} , called the model *represented by* X . We denote the corresponding mapping algebra $\mathcal{T}\{X\} := \mathcal{T}\{F_X\}$, with distinguished object X .

Definition 2.4. A \mathbf{Top}_* -enriched category \mathcal{T} is **left linear** if it satisfies the following conditions.

- (1) All mapping spaces $\mathcal{T}(A, B)$ are topological abelian groups, with the basepoint 0 being the additive identity.
- (2) Composition is left linear, i.e., satisfies $(x + x')y = xy + x'y$.

A **morphism** of left linear \mathbf{Top}_* -enriched categories is a \mathbf{Top}_* -enriched functor $F: \mathcal{S} \rightarrow \mathcal{T}$ such that for all objects A, B of \mathcal{S} , the induced map

$$F: \mathcal{S}(A, B) \rightarrow \mathcal{T}(FA, FB)$$

is a group homomorphism (i.e., preserves addition strictly).

A mapping algebra (\mathcal{T}, F) is **left linear** if \mathcal{T} is left linear and the following conditions hold.

- (1a) All spaces $F(A)$ are topological abelian groups, with the basepoint 0 being the additive identity.
- (2a) Composition in $\mathcal{T}\{F\}$ of the form

$$\mathcal{T}(A, B) \wedge F(A) \rightarrow F(B)$$

is left linear. Equivalently, the map in \mathbf{Top}_*

$$F: \mathcal{T}(A, B) \rightarrow \mathbf{Top}_*(FA, FB)$$

is a group homomorphism, where the right-hand side has pointwise addition in the target $F(B)$.

The mapping theory \mathcal{T} is **weakly bilinear** if it is left linear and the following condition holds:

- (3) For all objects A, B, Z of \mathcal{T} the map

$$\mathcal{T}(A \times B, Z) \xrightarrow{(i_A^*, i_B^*)} \mathcal{T}(A, Z) \times \mathcal{T}(B, Z)$$

is a trivial fibration, i.e., a Serre fibration and a weak equivalence. Here, $i_A: A \rightarrow A \times B$ and $i_B: B \rightarrow A \times B$ denote the inclusion maps given by $i_A = (1_A, 0)$ and $i_B = (0, 1_B)$. In particular, products in \mathcal{T} are weak coproducts.

Remark 2.5. We shall see that condition (3) implies that composition is right linear up to homotopy, which explains the terminology “weakly bilinear”.

2.2. The Eilenberg–MacLane mapping theory. In this section, we describe our motivational example of a mapping theory, which uses specific models of Eilenberg–MacLane spectra.

Notation 2.6. Let **Spec** denote the category of *Bousfield–Friedlander spectra*, viewed as a \mathbf{Top}_* -enriched category. Details are given in Appendix A.

Proposition 2.7. *Let A be an abelian group, and let $\Sigma^n HA$ denote the n -fold suspension of the Eilenberg–MacLane spectrum HA in the stable homotopy category \mathbf{HoSpec} . There exist spectra K_n^A of the homotopy type of $\Sigma^n HA$ such that the full subcategory $\mathcal{EM}(\mathbf{Ab})$ of **Spec** consisting of finite products*

$$K = K_{n_1}^{A_1} \times \dots \times K_{n_k}^{A_k}$$

is a weakly bilinear mapping theory.

Proof. See Appendix A. □

Notation 2.8. Fix a prime number p and let $\mathbb{F} := \mathbb{F}_p$ denote the field of order p . Let \mathcal{A} denote the mod p Steenrod algebra $H\mathbb{F}_p^* H\mathbb{F}_p$. Cohomology H^*X will denote mod p cohomology $H^*(X; \mathbb{F}_p)$, viewed as a *left* \mathcal{A} -module. We denote $K_n := K_n^{\mathbb{F}_p}$.

Definition 2.9. Let \mathcal{EM} denote the full subcategory of **Spec** consisting of finite products K as above, where all coefficient groups A_i are \mathbb{F}_p . This category \mathcal{EM} is also a weakly bilinear mapping theory.

Remark 2.10. Consider the full subcategory $\overline{\mathcal{EM}}$ of \mathbf{Spec} with objects the bounded below degree-wise finite products $K = \prod_i K_{n_i}$. This category $\overline{\mathcal{EM}}$ is also a weakly bilinear mapping theory. It appears notably in the context of $H\mathbb{F}_p$ -based Adams resolutions [9, §7].

The mapping theory \mathcal{EM} is a topological refinement of the Steenrod algebra \mathcal{A} . In fact, the category $\pi_0\mathcal{EM}$ of path components of \mathcal{EM} is equivalent to the opposite of the category of finitely generated free \mathcal{A} -modules, so that $\pi_0\mathcal{EM}$ is the theory of \mathcal{A} -modules.

Definition 2.11. Given a cofibrant spectrum X , the representable model F_X of the mapping theory \mathcal{EM} , given by

$$F_X(K) := \mathbf{Spec}(X, K)$$

is called the **topological cohomology** of X with mod p coefficients. We write $\mathcal{EM}\{X\}$ for the mapping algebra (\mathcal{EM}, F_X) , as in Remark 2.3. Note that $\pi_0\mathcal{EM}\{X\}$ encodes the primary cohomology of X as an \mathcal{A} -module, given by

$$\pi_0 F_X(K_n) = [X, K_n] = H^n X.$$

A morphism in $\pi_0\mathcal{EM}$ is a primary cohomology operation. By applying the fundamental groupoid functor $\Pi_{(1)}$ to \mathcal{EM} , we obtain a theory $\Pi_{(1)}\mathcal{EM}$ enriched in groupoids. Morphisms in $\Pi_{(1)}\mathcal{EM}(K, L)$ are called *secondary cohomology operations*.

In [8], it was shown how the classical Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*X, \mathbb{F}_p) \Rightarrow \pi_{t-s} X_p^\wedge$$

can be derived from $\mathcal{EM}\{X\}$, more specifically, how the Postnikov section $P_n\mathcal{EM}\{X\}$ determines the spectral sequence up to the E_{n+2} term. In particular, the *secondary cohomology* $\Pi_{(1)}\mathcal{EM}\{X\}$ of the spectrum X determines the E_3 term of the spectral sequence. The algebraic structure of the theory $\Pi_{(1)}\mathcal{EM}$ is studied in more detail in [10].

3. HIGHER DISTRIBUTIVITY

3.1. Cubes in a space. In this subsection, we fix some notation about cubes in a space or a topologically enriched category.

Definition 3.1. Let X be a topological space.

An *n -cube* in X is a map $\gamma: I^n \rightarrow X$, where $I = [0, 1]$ is the unit interval. For example, a 0-cube in X is a point of X , and a 1-cube in X is a path in X . In this case, we also denote γ as an arrow $\gamma: \gamma(0) \rightarrow \gamma(1)$.

An *n -track* in X is a homotopy class, relative to the boundary ∂I^n , of an n -cube. If $\gamma: I^n \rightarrow X$ is an n -cube in X , denote by $\{\gamma\}$ the corresponding n -track in X , namely the homotopy class of γ rel ∂I^n .

Definition 3.2. Let X be a pointed space, with basepoint $0 \in X$. The constant map $0: I^n \rightarrow X$ with value $0 \in X$ is called the **trivial n -cube**.

The equality $I^{m+n} = I^m \times I^n$ allows us to define an operation on cubes.

Definition 3.3. Let $\mu: X \times X' \rightarrow X''$ be a composition map in a \mathbf{Top}_* -enriched category \mathcal{C} . For $m, n \geq 0$, consider cubes

$$\begin{aligned} a: I^m &\rightarrow X \\ b: I^n &\rightarrow X'. \end{aligned}$$

The \otimes -**composition** of a and b is the $(m+n)$ -cube $a \otimes b$ defined as the composite

$$(3.1) \quad a \otimes b: I^{m+n} = I^m \times I^n \xrightarrow{a \times b} X \times X' \xrightarrow{\mu} X''.$$

For $m = n$, the **pointwise composition** of a and b is the n -cube defined as the composite

$$(3.2) \quad ab: I^n \xrightarrow{(a,b)} X \times X' \xrightarrow{\mu} X''.$$

The pointwise composition is the restriction of the \otimes -composition along the diagonal:

$$\begin{array}{ccc} I^n & \xrightarrow{\Delta} & I^n \times I^n \xrightarrow{a \otimes b} X'' \\ & \searrow \quad \nearrow & \\ & ab & \end{array}$$

Similarly, let X be a topological abelian group. The **external addition** of cubes $a: I^m \rightarrow X$ and $b: I^n \rightarrow X$ is the $(m+n)$ -cube $a \oplus b$ defined as the composite

$$(3.3) \quad a \oplus b: I^{m+n} = I^m \times I^n \xrightarrow{a \times b} X \times X \xrightarrow{+} X.$$

For $m = n$, the **pointwise addition** of a and b is the n -cube defined as the composite

$$(3.4) \quad a + b: I^n \xrightarrow{(a,b)} X \times X \xrightarrow{+} X.$$

The pointwise addition is the restriction of the exterior addition along the diagonal:

$$\begin{array}{ccc} I^n & \xrightarrow{\Delta} & I^n \times I^n \xrightarrow{a \oplus b} X \\ & \searrow \quad \nearrow & \\ & a+b & \end{array}$$

As an abuse of notation, we will also write $xy := x \otimes y$ and $x + y := x \oplus y$ if $\deg(x) = 0$ or $\deg(y) = 0$ holds.

Lemma 3.4. *Let \mathcal{T} be a left linear \mathbf{Top}_* -enriched category. Then the \otimes -composition with a 0-cube is left linear with respect to the external addition. More precisely, consider cubes in a mapping space in \mathcal{T}*

$$\begin{cases} a: I^m \rightarrow \mathcal{T}(A, B) \\ b: I^n \rightarrow \mathcal{T}(A, B) \end{cases}$$

and a map $x: X \rightarrow A$. Then the equality

$$(a \oplus b)x = (ax) \oplus (bx)$$

holds, where both sides are $(m+n)$ -cubes in $\mathcal{T}(X, B)$.

Proof. In the diagram of spaces

$$\begin{array}{ccccc} I^m \times I^n \times I^0 & \xrightarrow{a \times b \times x} & \mathcal{T}(A, B) \times \mathcal{T}(A, B) \times \mathcal{T}(X, A) & \xrightarrow{(+_B)_* \times \text{id}} & \mathcal{T}(A, B) \times \mathcal{T}(X, A) \\ \parallel & & \downarrow \text{id} \times \text{id} \times \Delta & & \searrow \mu \\ I^{m+n} & & \mathcal{T}(A, B) \times \mathcal{T}(A, B) \times \mathcal{T}(X, A) \times \mathcal{T}(X, A) & & \mathcal{T}(X, B) \\ \parallel & & \downarrow \text{permute} & & \nearrow (+_B)_* \\ I^m \times I^0 \times I^n \times I^0 & \xrightarrow{a \times x \times b \times x} & \mathcal{T}(A, B) \times \mathcal{T}(X, A) \times \mathcal{T}(A, B) \times \mathcal{T}(X, A) & \xrightarrow{\mu \times \mu} & \mathcal{T}(X, B) \times \mathcal{T}(X, B) \end{array}$$

the left part commutes by construction, and the right part commutes by left linearity of \mathcal{T} . The composite along the top is $(a \oplus b) \otimes x$, whereas the composite along the bottom is $(a \otimes x) \oplus (b \otimes x)$. \square

3.2. Definition of higher distributivity. Before defining higher distributivity in general, let us look at some low-dimensional cases.

Definition 3.5. Let \mathcal{T} be a left linear \mathbf{Top}_* -enriched category, or a left linear mapping algebra $\mathcal{T}\{\star\}$ with distinguished object \star . Then \mathcal{T} is called **1-distributive** if for all $a, x, y \in \mathcal{T}$, there is a path

$$a(x + y) \xrightarrow{\varphi_a^{x,y}} ax + ay.$$

in \mathcal{T} . A choice of such paths for $a, x, y \in \mathcal{T}$ is denoted $\varphi^1 = \{\varphi_a^{x,y} \mid a, x, y \in \mathcal{T}\}$ and is called a **1-distributor** for \mathcal{T} . Per Notation 2.1, here we mean for all $a, x, y \in \mathcal{T}$ such that $a(x + y)$ is defined. Also, φ^1 is required to be continuous in the inputs a, x, y . More precisely, for all objects X, A, B of \mathcal{T} , the map

$$\begin{aligned} \mathcal{T}(A, B) \times \mathcal{T}(X, A)^2 &\xrightarrow{\varphi^1} \mathcal{T}(X, B)^I \\ (a, x, y) &\longmapsto \varphi_a^{x,y} \end{aligned}$$

is continuous.

Next, \mathcal{T} is called **2-distributive** if it admits a 1-distributor φ^1 such that for all $a, x, y, z \in \mathcal{T}$, the map $\partial I^2 \rightarrow \mathcal{T}$ defined as in the diagram of paths

$$\begin{array}{ccccc} a(x + y) + az & \xrightarrow{\varphi_a^{x,y} + az} & ax + ay + az \\ \uparrow \varphi_a^{x+y,z} & & \uparrow ax + \varphi_a^{y,z} \\ a(x + y + z) & \xrightarrow{\varphi_a^{x,y+z}} & ax + a(y + z). \end{array}$$

admits an extension $\varphi_a^{x,y,z} : I^2 \rightarrow \mathcal{T}$. A choice of such 2-cubes for $a, x, y, z \in \mathcal{T}$ is denoted

$$\varphi^2 = \{\varphi_a^{x,y,z} \mid a, x, y, z \in \mathcal{T}\}$$

and is called a **2-distributor** for \mathcal{T} , **based** on the 1-distributor φ^1 . As before, the 2-distributor φ^2 is required to be continuous in the inputs $a, x, y, z \in \mathcal{T}$.

By convention, define a **0-distributor** as the collection of 0-cubes $\varphi_a^x = ax$.

Definition 3.6. A left linear \mathbf{Top}_* -enriched category \mathcal{T} is called **n -distributive** if there are collections of cubes $\varphi^0, \varphi^1, \dots, \varphi^n$, where

$$\varphi^m = \{\varphi_a^{x_0, \dots, x_m} \mid a, x_0, \dots, x_m \in \mathcal{T}\}$$

is a collection of m -cubes $\varphi_a^{x_0, \dots, x_m} : I^m \rightarrow \mathcal{T}$, satisfying the following:

- φ^0 is a 0-distributor, i.e., the collection of 0-cubes $\varphi_a^x = ax$.

- For $1 \leq m \leq n$, the following boundary conditions hold:

$$(3.5) \quad \begin{cases} \varphi_a^{x_0, \dots, x_m}(t_1, \dots, \overbrace{0}^{t_j}, \dots, t_m) = \varphi_a^{x_0, \dots, x_{j-1}+x_j, \dots, x_m}(t_1, \dots, \widehat{t_j}, \dots, t_m) \\ \varphi_a^{x_0, \dots, x_m}(t_1, \dots, \overbrace{1}^{t_j}, \dots, t_m) = \varphi_a^{x_0, \dots, x_{j-1}}(t_1, \dots, t_{j-1}) \oplus \varphi_a^{x_j, \dots, x_m}(t_{j+1}, \dots, t_m). \end{cases}$$

Such a collection φ^n of n -cubes in \mathcal{T} is called an **n -distributor** for \mathcal{T} , **based** on the $(n-1)$ -distributor φ^{n-1} . The n -distributor φ^n is required to be continuous in the inputs $a, x_0, \dots, x_n \in \mathcal{T}$. More precisely, for all objects X, A, B of \mathcal{T} , the map

$$\begin{aligned} \mathcal{T}(A, B) \times \mathcal{T}(X, A)^{n+1} &\xrightarrow{\varphi^n} \mathcal{T}(X, B)^{I^n} \\ (a, x_0, \dots, x_n) &\longmapsto \varphi_a^{x_0, \dots, x_n} \end{aligned}$$

is continuous. Note that the case $n = 2$ agrees with Definition 3.5.

The case $n = \infty$ is allowed: An **∞ -distributor** for \mathcal{T} is a sequence $\{\varphi^m\}_{m \geq 0}$ of families of cubes satisfying the above conditions, for all $m \geq 0$.

This definition is closely related to the notion of A_∞ morphisms. The connection is described more precisely in Section 6.2.

Remark 3.7. In the data of an n -distributor, we only retain the n -cubes φ^n , since the lower dimensional cubes φ^k (for $0 \leq k \leq n-1$) are determined by the boundary condition

$$\begin{aligned} \varphi_a^{x_0, \dots, x_{m-1}, 0}(t_1, \dots, t_{m-1}, 0) &= \varphi_a^{x_0, \dots, x_{m-1}+0}(t_1, \dots, t_{m-1}) \\ &= \varphi_a^{x_0, \dots, x_{m-1}}(t_1, \dots, t_{m-1}). \end{aligned}$$

The following theorem will be proved in Section 4.

Theorem 3.8. *A weakly bilinear mapping theory is ∞ -distributive.*

3.3. Inductive construction of distributors. Applying the boundary conditions of φ^n in Equation (3.5) repeatedly, one can find the restriction $\varphi^n|_C: C \rightarrow \mathcal{T}$ to any face $C \subseteq I^n$ of dimension less than n . We now describe this formula explicitly.

Notation 3.9. Let $n \geq 1$. The cells (or subcubes) of the cube I^n consist of the subsets $C \subseteq I^n$ of the form

$$X_1 \times \dots \times X_n \subseteq I^n$$

where each X_i is $\{0\}$ or $\{1\}$ or $I = [0, 1]$. These cells are in bijection with functions $\sigma: \{1, \dots, n\} \rightarrow \{0, 1, I\}$, which we call **codes** for convenience, and sometimes write as a sequence of values $(\sigma(1), \dots, \sigma(n))$. Denote by C_σ the cell corresponding to σ , with its inclusion map $\text{inc}_\sigma: C_\sigma \hookrightarrow I^n$.

The subcube C_σ has dimension $|\sigma^{-1}(I)|$, the number of free coordinates. Because of this, we also denote $\dim \sigma := |\sigma^{-1}(I)|$. The entire cube is $I^n = C_{(I, \dots, I)}$, while its boundary is the union of proper faces

$$\partial I^n = \bigcup_{\substack{\sigma \in \{0, 1, I\}^n \\ \sigma \neq (I, \dots, I)}} C_\sigma.$$

The vertices of the cube I^n will parametrize different ways of going from $a(x_0 + \dots + x_n)$ to $ax_0 + \dots + ax_n$ by distributing the product over the sums. These expressions contain

n symbols $+$, and we will interpret the value $\sigma(i)$ as telling whether the i^{th} symbol $+$ has been brought outside, with 1 or 0 meaning yes or no, respectively. For example, $(1, 0, 0, 1)$ corresponds to $ax_0 + a(x_1 + x_2 + x_3) + ax_4$.

Example 3.10. Consider the case $n = 2$. The subcubes of I^2 are labeled by codes $\sigma \in \{0, 1, I\}^2$ as illustrated in Figure 3.1.

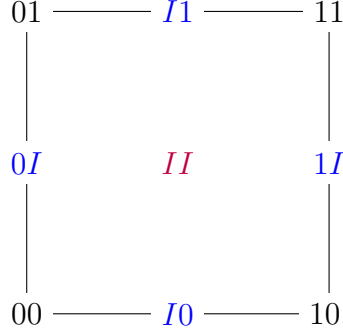


FIGURE 3.1. The subcubes of the cube I^2 .

The subcubes are assigned distributors as in Figure 3.2.

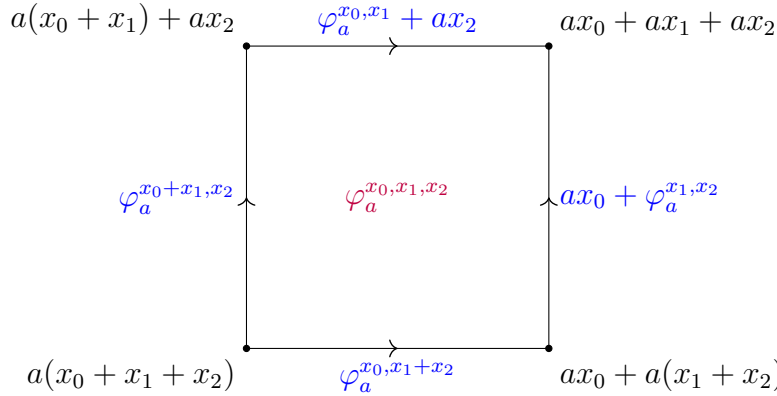


FIGURE 3.2. The distributors assigned to faces of a 2-cube.

If a 1-distributor φ^1 is given, then this defines the obstruction map $\mathcal{O}(\varphi^1): \partial I^2 \rightarrow \mathcal{T}$ of Definition 3.14.

Note that the product ax_i is itself the 0-distributor $\varphi_a^{x_i}$. Also note that the pointwise sum of 0-cubes $ax_0 + ax_1$ is also an external sum $ax_0 \oplus ax_1$, and the expression $\varphi_a^{x_0, x_1} + ax_2$ is shorthand notation for the external sum $\varphi_a^{x_0, x_1} \oplus ax_2$.

Definition 3.11. Let $n \geq 1$ and let φ^m be an m -distributor for \mathcal{T} . Let $C_\sigma \subseteq I^n$ be a proper subcube of dimension $\dim \sigma \leq m$. The **face formula** associates to each $a, x_0, \dots, x_n \in \mathcal{T}$ (such that the expression $a(x_0 + \dots + x_n)$ is defined) a map

$$\varphi^m[\sigma]_a^{x_0, \dots, x_n} : C_\sigma \rightarrow \mathcal{T}$$

as follows.

Step 1: By convention, extend the code σ by $\sigma(0) = 1$. Let

$$\{0, 1, \dots, n\} = J_0 \sqcup J_1 \sqcup \dots \sqcup J_t$$

be the partition into intervals satisfying

$$(3.6) \quad \sigma|_{J_k}(i) = \begin{cases} 1 & \text{if } i = \min J_k \\ 0 \text{ or } I & \text{if } i \neq \min J_k. \end{cases}$$

In particular, $t = |\sigma^{-1}(1)|$ is the number of 1's in the code. We define

$$\varphi^m[\sigma] := \varphi^m[\sigma|_{J_0}] \oplus \varphi^m[\sigma|_{J_1}] \oplus \dots \oplus \varphi^m[\sigma|_{J_t}].$$

Step 2: For an interval of integers J satisfying Equation (3.6), let

$$J = K_0 \sqcup K_1 \sqcup \dots \sqcup K_d$$

be the partition into intervals satisfying

$$\sigma|_{K_l}(i) = \begin{cases} 1 \text{ or } I & \text{if } i = \min K_l \\ 0 & \text{if } i \neq \min K_l. \end{cases}$$

In particular, $d = |\sigma^{-1}(I) \cap J|$ is the number of I 's in $\sigma|_J$. We define

$$\varphi^m[\sigma|_J] := \varphi_a^{x_{K_0}, x_{K_1}, \dots, x_{K_d}}$$

where we denote $x_K := \sum_{k \in K} x_k$ for any set of integers K .

Example 3.12. With $n = 8$ and the code $\sigma = 0I11I00I$, we have:

	$\overbrace{\hspace{1.5cm}}^{J_0}$			$\overbrace{\hspace{1.5cm}}^{J_1}$		$\overbrace{\hspace{1.5cm}}^{J_2}$			
i	0	1	2	3	4	5	6	7	8
$\sigma(i)$	1	0	I	1	1	I	0	0	I

Next, $J_2 = \{4, 5, 6, 7, 8\}$ is partitioned as

	$\overbrace{\hspace{1.5cm}}^{K_0}$		$\overbrace{\hspace{1.5cm}}^{K_1}$		$\overbrace{\hspace{1.5cm}}^{K_2}$		
i	4	5	6	7	8		
$\sigma(i)$	1	I	0	0	I		

and likewise $J_0 = \{0, 1, 2\} = \{0, 1\} \sqcup \{2\}$. This yields

$$\begin{aligned} \varphi^m[\sigma] &= \varphi^m[\sigma|_{J_0}] \oplus \varphi^m[\sigma|_{J_1}] \oplus \varphi^m[\sigma|_{J_2}] \\ &= \varphi_a^{x_0+x_1, x_2} \oplus \varphi_a^{x_3} \oplus \varphi_a^{x_4, x_5+x_6+x_7, x_8} \end{aligned}$$

which is defined as long as $m \geq 3 = \dim \sigma$ holds.

Lemma 3.13. *A (continuous) collection φ^n of cubes $I^n \rightarrow \mathcal{T}$ is an n -distributor if and only if it satisfies $\varphi^n|_{C_\sigma} = \varphi^n[\sigma]$ for every subcube $C_\sigma \subseteq I^n$.*

Definition 3.14. Let φ^{n-1} be an $(n-1)$ -distributor for \mathcal{T} . The **obstruction to n -distributivity** is the collection of maps

$$\mathcal{O}(\varphi^{n-1})_a^{x_0, \dots, x_n} : \partial I^n \rightarrow \mathcal{T}$$

defined by their restriction to each face $C_\sigma \subset \partial I^n$:

$$\mathcal{O}(\varphi^{n-1})|_{C_\sigma} := \varphi^{n-1}[\sigma].$$

It follows from the face formula that $\varphi^{n-1}[\sigma]$ and $\varphi^{n-1}[\sigma']$ agree on the intersection $C_\sigma \cap C_{\sigma'}$, so that the map $\mathcal{O}(\varphi^{n-1}): \partial I^n \rightarrow \mathcal{T}$ is well-defined.

Example 3.15. Consider the case $n = 3$. The subcubes of I^3 are labeled by codes $\sigma \in \{0, 1, I\}^3$ as illustrated in Figure 3.3.

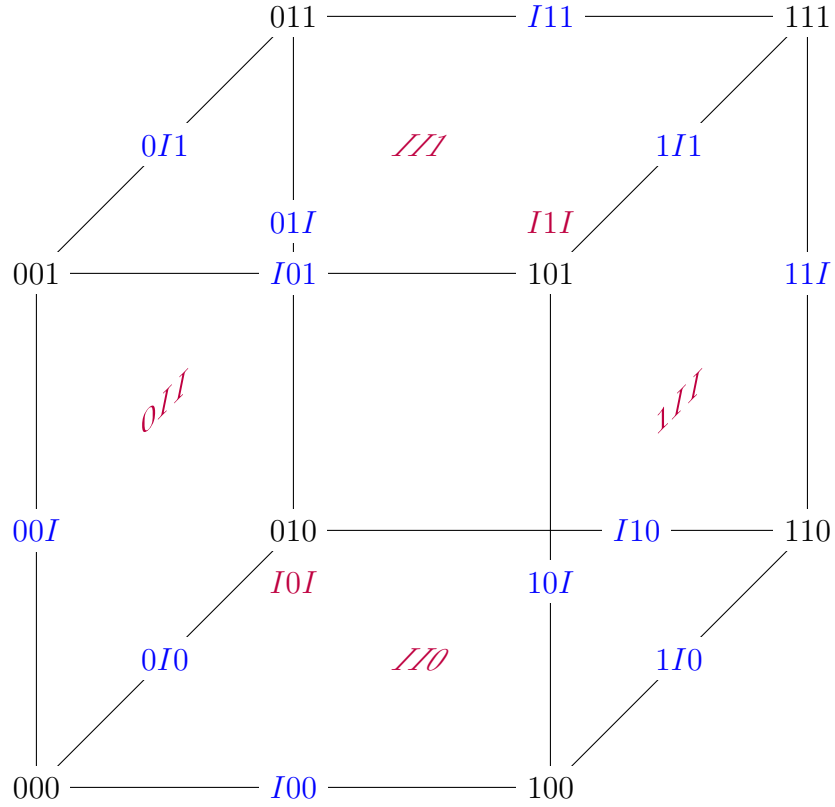
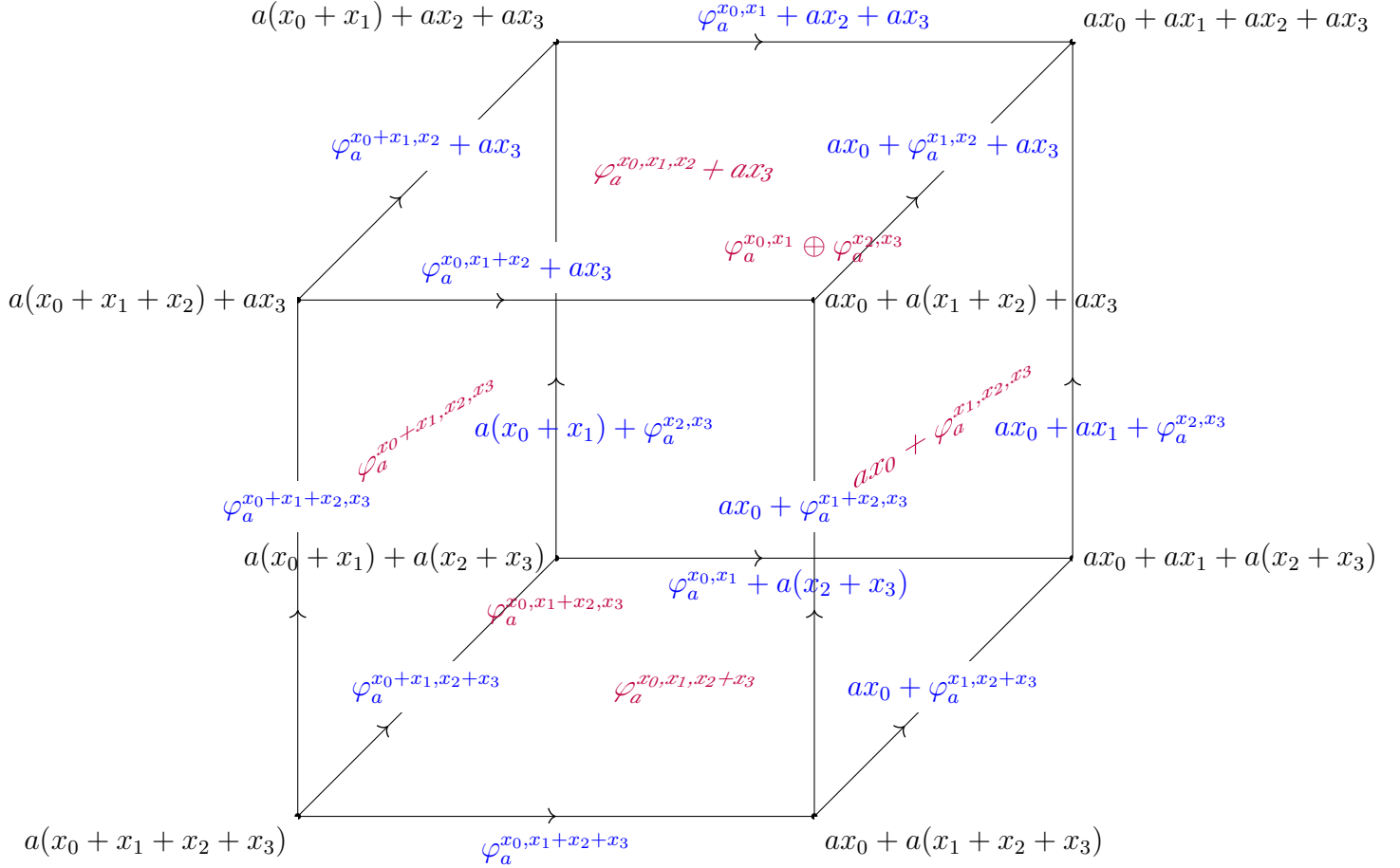


FIGURE 3.3. The subcubes of the cube I^3 .

The subcubes are assigned distributors as in Figure 3.4.

FIGURE 3.4. The obstruction map $\mathcal{O}(\varphi^2): \partial I^3 \rightarrow \mathcal{T}$.

We can now reinterpret higher distributivity as an inductive construction. The following lemma could be taken as an alternate to Definition 3.6.

Lemma 3.16. *Let $n \geq 1$ and let \mathcal{T} be a left linear \mathbf{Top}_* -enriched category. A (continuous) family of n -cubes in \mathcal{T}*

$$\varphi^n = \{\varphi_a^{x_0, \dots, x_n} \mid a, x_0, \dots, x_n \in \mathcal{T}\}$$

and is an n -distributor for \mathcal{T} if and only if φ^{n-1} is an $(n-1)$ -distributor for \mathcal{T} , and for all $a, x_0, x_1, \dots, x_n \in \mathcal{T}$, the n -cube $\varphi_a^{x_0, \dots, x_n}: I^n \rightarrow \mathcal{T}$ extends the obstruction map from Definition 3.14:

$$\varphi_a^{x_0, \dots, x_n}|_{\partial I^n} = \mathcal{O}(\varphi^{n-1})_a^{x_0, \dots, x_n}: \partial I^n \rightarrow \mathcal{T}.$$

In shorter notation: $\varphi^n|_{\partial I^n} = \mathcal{O}(\varphi^{n-1})$.

Moreover, a sequence $\{\varphi^0, \varphi^1, \varphi^2, \dots\}$ is an ∞ -distributor for \mathcal{T} if and only if each φ^n is an n -distributor based on φ^{n-1} .

Remark 3.17. The compatibility condition $\varphi^n|_{C_\sigma} = \varphi^n[\sigma]$ as in Lemma 3.13 can be described more explicitly. For every $n \geq 0$, all inputs a, x_0, \dots, x_n (which we will omit from the notation),

and every subcube $C_\sigma \subseteq I^n$, the diagram

$$(3.7) \quad \begin{array}{ccc} C_\sigma & \xrightarrow{\varphi[\sigma]} & \mathcal{T} \\ \text{inc}_\sigma \downarrow & \nearrow \varphi^n & \\ I^n & & \end{array}$$

commutes. Parametrizing the subcubes $C_\sigma \subseteq I^n$ by codes $\sigma: \{1, \dots, n\} \rightarrow \{0, 1, I\}$, consider the partition into intervals

$$\{0, 1, \dots, n\} = J_0 \sqcup J_1 \sqcup \dots \sqcup J_t$$

as in Definition 3.11. To match our convention $\sigma(0) = 1$, let us view the cube I^n as

$$I^n \cong \{1\} \times I^n \hookrightarrow I^{n+1},$$

with 0th coordinate taking value 1. Then the Diagram (3.7) can be written more explicitly as:

$$(3.8) \quad \begin{array}{c} \xrightarrow{\varphi[\sigma|_{J_0}] \oplus \dots \oplus \varphi[\sigma|_{J_t}]} \\ I^{\dim \sigma|_{J_0}} \times \dots \times I^{\dim \sigma|_{J_t}} \xlongequal{\quad} I^{\dim \sigma} \xlongequal{\quad} C_\sigma \xrightarrow{\varphi[\sigma]} \mathcal{T} \\ \downarrow \text{inc}_{\sigma|_{J_0}} \times \dots \times \text{inc}_{\sigma|_{J_t}} \quad \downarrow \text{inc}_\sigma \quad \nearrow \varphi^n \\ \{1\} \times I^n \xlongequal{\quad} I^n \\ \downarrow \\ I^{|J_0|} \times \dots \times I^{|J_t|} \xlongequal{\quad} I^{n+1}. \end{array}$$

3.4. Combinatorics of cubes and their faces. In this optional section, we provide an alternate description of the combinatorics of cubes and their faces, following [4, §I.2].

Notation 3.18. Let Δ denote the simplicial indexing category, whose objects are the finite ordinals $[\mathbf{n}] = \{0, 1, \dots, n\}$, for $n \geq 0$, and maps $\alpha: [\mathbf{m}] \rightarrow [\mathbf{n}]$ are order-preserving functions.

Definition 3.19. Let $n \geq 1$. A **string** in $[\mathbf{n} + 1]$ is a tuple $\lambda = (\alpha_1, \dots, \alpha_r)$ of injective maps $\alpha_i: [\mathbf{n}_i] \rightarrow [\mathbf{n} + 1]$ in Δ satisfying the following conditions:

$$\begin{cases} \alpha_1(0) = 0 \\ \alpha_i(n_i) = \alpha_{i+1}(0) \quad \text{for } i = 1, \dots, r-1 \\ \alpha_r(n_r) = n+1. \end{cases}$$

We can also denote $\alpha_i: [\mathbf{n}_i] \rightarrow [\mathbf{n} + 1]$ by the sequence of its values $(\alpha_i(0), \dots, \alpha_i(n_i))$, and denote a string λ by writing the sequences of each α_i one after the other. For instance, $(0, 3)(3, 4, 7)(7, 8)$ represents a string in [8].

Given a string $\lambda = (\alpha_1, \dots, \alpha_r)$ in $[\mathbf{n} + 1]$ define the following subsets of $[\mathbf{n} + 1]$:

$$\begin{cases} \lambda_0 := [\mathbf{n} + 1] \setminus \bigcup_{i=1}^r \alpha_i([\mathbf{n}_i]) \\ \lambda_I := \bigcup_{i=1}^r \alpha_i([\mathbf{n}_i] \setminus \{0, n_i\}) \\ \lambda_1 := \{\alpha_1(0), \alpha_2(0), \dots, \alpha_r(0)\}. \end{cases}$$

Note that these three sets form a partition $[\mathbf{n}] = \lambda_0 \sqcup \lambda_I \sqcup \lambda_1$, with $0 \in \lambda_1$ by definition. Now associate to λ a code $\tilde{\lambda} \in \{0, 1, I\}^n$ defined by

$$\tilde{\lambda}(i) = \begin{cases} 0 & \text{if } i \in \lambda_0 \\ I & \text{if } i \in \lambda_I \\ 1 & \text{if } i \in \lambda_1. \end{cases}$$

One readily checks the following claim.

Proposition 3.20. (1) *The association $\lambda \mapsto \tilde{\lambda}$ described above defines a bijection from the set of strings in $[\mathbf{n} + \mathbf{1}]$ to the set $\{0, 1, I\}^n$.*

(2) *Via this bijection, the face formula $\varphi^{n-1}[\lambda]: C_\lambda \rightarrow \mathcal{T}$ from Definition 3.11 is given by*

$$\varphi^{n-1}[\lambda] = \varphi[\alpha_1] \oplus \cdots \oplus \varphi[\alpha_r]$$

where given an injective map $\alpha: [\mathbf{m}] \rightarrow [\mathbf{n} + \mathbf{1}]$ in Δ , we define the map $\varphi[\alpha]: I^{m-1} \rightarrow \mathcal{T}$ by the formula

$$\varphi[\alpha] = \varphi_a^{\sum_{i=\alpha(0)}^{\alpha(1)-1} x_i, \sum_{i=\alpha(1)}^{\alpha(2)-1} x_i, \dots, \sum_{i=\alpha(m-1)}^{\alpha(m)-1} x_i}.$$

Hence, strings in $[\mathbf{n} + \mathbf{1}]$ provide an alternate way of labeling the cells of the cube I^n .

Example 3.21. Consider the string $\lambda = (0, 3)(3, 4, 7)(7, 8)$ in $[\mathbf{8}]$. Its associated subsets are:

$$\begin{cases} \lambda_0 = \{1, 2, 5, 6\} \\ \lambda_I = \{4\} \\ \lambda_1 = \{0, 3, 7\} \end{cases}$$

and the associated code is $\tilde{\lambda} = 001I001$. The associated face formula is

$$\varphi^1[\lambda] = \varphi_a^{x_0+x_1+x_2} \oplus \varphi_a^{x_3, x_4+x_5+x_6} \oplus \varphi_a^{x_7}.$$

Let us redo the example directly, without going through the associated code $\tilde{\lambda}$. The string $\lambda = (\alpha_1, \alpha_2, \alpha_3)$ consists of the injective maps

$$\begin{cases} \alpha_1 = (0, 3): [\mathbf{1}] \rightarrow [\mathbf{8}] \\ \alpha_2 = (3, 4, 7): [\mathbf{2}] \rightarrow [\mathbf{8}] \\ \alpha_3 = (7, 8): [\mathbf{1}] \rightarrow [\mathbf{8}] \end{cases}$$

which define the maps

$$\begin{cases} \varphi[(0, 3)] = \varphi_a^{x_0+x_1+x_2} \\ \varphi[(3, 4, 7)] = \varphi_a^{x_3, x_4+x_5+x_6} \\ \varphi[(7, 8)] = \varphi_a^{x_7}. \end{cases}$$

This in turn yields

$$\varphi^1[\lambda] = \varphi_a^{x_0+x_1+x_2} \oplus \varphi_a^{x_3, x_4+x_5+x_6} \oplus \varphi_a^{x_7}.$$

Remark 3.22. Using strings $\lambda = (\alpha_1, \dots, \alpha_r)$ in $[\mathbf{n} + \mathbf{1}]$ to parametrize the subcubes $C_\lambda \subseteq I^n$, Diagram (3.8) becomes

$$(3.9) \quad \begin{array}{c} \xrightarrow{\varphi[\alpha_1] \oplus \dots \oplus \varphi[\alpha_r]} \\ I^{n_1-1} \times \dots \times I^{n_r-1} \xrightarrow{\quad} I^{\dim \lambda} \xrightarrow{\quad} C_\lambda \xrightarrow{\varphi[\lambda]} \mathcal{T} \\ \downarrow \text{inc}_{\alpha_1} \times \dots \times \text{inc}_{\alpha_r} \quad \downarrow \quad \downarrow \text{inc}_\lambda \quad \nearrow \varphi^n \\ \{1\} \times I^n \xrightarrow{\quad} I^n \\ \downarrow \\ I^{\alpha_1(n_1)-\alpha_1(0)} \times \dots \times I^{\alpha_r(n_r)-\alpha_r(0)} \xrightarrow{\quad} I^{n+1}. \end{array}$$

4. GOOD DISTRIBUTORS

In this section, we describe additional conditions on a distributor that will be satisfied in a weakly bilinear mapping theory.

For every $a, x_0, \dots, x_n \in \mathcal{T}$, the two maps being compared in the distributivity equation, namely $a(x_0 + \dots + x_n)$ and $ax_0 + \dots + ax_n$, factor through $(x_0, \dots, x_n): X \rightarrow A^{n+1}$, as illustrated in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{(x_0, \dots, x_n)} & A^{n+1} & \xrightarrow{a^{n+1}} & B^{n+1} \\ & \searrow x_0 + \dots + x_n & \downarrow +_A & & \downarrow +_B \\ & & A & \xrightarrow{a} & B. \end{array}$$

Hence, for fixed $a \in \mathcal{T}$, there is a universal case to consider, with $(x_0, \dots, x_n) = \text{id}_{A^{n+1}}$. In other words, take $x_i = p_i: A^{n+1} \rightarrow A$, where the maps $p_0, \dots, p_n: A^{n+1} \rightarrow A$ denote the projections onto the factors.

Also, the equalities $a(x + 0) = a(0 + x) = ax$ hold strictly in \mathcal{T} . More generally, given inputs x_0, \dots, x_n with $x_i = 0$ for $i \neq k$, then the equality $a(x_0 + \dots + x_n) = ax_k$ holds. In other words, for all maps $a: A \rightarrow B$ and $x: X \rightarrow A$, and index $0 \leq k \leq n$, consider the diagram in \mathcal{T}

$$\begin{array}{ccccc} X & \xrightarrow{(0, \dots, x^{k\text{th}}, \dots, 0)} & A^{n+1} & \xrightarrow{a^{n+1}} & B^{n+1} \\ \downarrow x & \nearrow i_k & \downarrow +_A & & \downarrow +_B \\ A & \xrightarrow{\quad} & A & \xrightarrow{a} & B. \end{array}$$

The left half commutes, but the right square does *not* commute. However, the large rectangle does commute, as both composites are equal to $ax: X \rightarrow B$.

These observations lead to the following:

Definition 4.1. An n -distributor φ^n is **good** if it satisfies the following properties.

(1) (*Universality*) For all $a, x_0, \dots, x_n \in \mathcal{T}$, the equality

$$(4.1) \quad \varphi_a^{x_0, \dots, x_n} = \varphi_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n)$$

holds. Both sides are n -cubes in $\mathcal{T}(X, B)$.

- (2) (*Wedge condition*) For all maps $a: A \rightarrow B$ and $x: X \rightarrow A$ in \mathcal{T} , and index $0 \leq k \leq n$, the cube

$$\varphi_a^{0,\dots,x,\dots,0}: I^n \rightarrow \mathcal{T}(X, B)$$

is the constant n -cube at ax .

Remark 4.2. If \mathcal{T} happened to come from an ambient model category \mathcal{C} , then the restriction

$$(i_0^*, \dots, i_n^*): \mathcal{T}(A^{n+1}, B) \rightarrow \mathcal{T}(A, B)^{n+1} \cong \mathcal{C}(\bigvee_{i=0}^n A, B)$$

corresponds to restriction along the inclusion of the wedge $\bigvee_{i=0}^n A \hookrightarrow A^{n+1}$. The wedge condition says that no correction is needed when we restrict to the wedge.

Note that if φ^n is good, then φ^{n-1} is automatically good as well. Also note that the 0-distributor φ^0 is good, trivially.

Lemma 4.3. *Let φ^n be an n -distributor satisfying the universality condition. Then φ^n satisfies the wedge condition if and only if for every map $a: A \rightarrow B$ in \mathcal{T} , the composite*

$$I^n \xrightarrow{\varphi_a^{p_0, \dots, p_n}} \mathcal{T}(A^{n+1}, B) \xrightarrow{(i_0^*, \dots, i_n^*)} \mathcal{T}(A, B)^{n+1}$$

is the constant n -cube at (a, \dots, a) .

Proof. The wedge condition says that for all $x: X \rightarrow A$ and all index $0 \leq k \leq n$, the map

$$\varphi_a^{0,\dots,x,\dots,0}: I^n \rightarrow \mathcal{T}(X, B)$$

is the constant n -cube cst_{ax} at $ax \in \mathcal{T}(X, B)$. Since φ^n satisfies universality, we have

$$\begin{aligned} \varphi_a^{0,\dots,x,\dots,0} &= \varphi_a^{p_0, \dots, p_n} \otimes (0, \dots, x, \dots, 0) \\ &= \varphi_a^{p_0, \dots, p_n} \otimes (i_k x) \\ &= \varphi_a^{p_0, \dots, p_n} \otimes i_k \otimes x. \end{aligned}$$

This cube is the constant n -cube at ax for all x if and only if

$$\varphi_a^{p_0, \dots, p_n} \otimes i_k: I^n \rightarrow \mathcal{T}(A, B)$$

is the constant n -cube at $a \in \mathcal{T}(A, B)$. Indeed, the “only if” direction is a special case $x = 1_A: A \rightarrow A$, whereas the “if” direction follows from the equality of n -cubes $\text{cst}_a \otimes x = \text{cst}_{ax}$. To conclude, the equality of n -cubes

$$\varphi_a^{p_0, \dots, p_n} \otimes i_k = i_k^* \varphi_a^{p_0, \dots, p_n} = \text{cst}_a$$

holds for all index $0 \leq k \leq n$ if and only if the equality

$$(i_0^*, \dots, i_n^*) \varphi_a^{p_0, \dots, p_n} = (\text{cst}_a, \dots, \text{cst}_a) = \text{cst}_{(a, \dots, a)}$$

holds. □

Lemma 4.4. *If an $(n-1)$ -distributor φ^{n-1} satisfies the universality property 4.1 (1), then the obstruction to n -distributivity $\mathcal{O}(\varphi^{n-1})$ satisfies the following analogous property: For all $a, x_0, \dots, x_n \in \mathcal{T}$, the equality*

$$\mathcal{O}(\varphi^{n-1})_a^{x_0, \dots, x_n} = \mathcal{O}(\varphi^{n-1})_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n)$$

holds. Both sides are maps $\partial I^n \rightarrow \mathcal{T}(X, B)$.

Proof. Let us show that both sides agree when restricted to any face $C_\sigma \subseteq \partial I^n$. The right-hand side is:

$$\begin{aligned} & (\mathcal{O}(\varphi^{n-1})_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n))|_{C_\sigma} \\ &= (\mathcal{O}(\varphi^{n-1})_a^{p_0, \dots, p_n})|_{C_\sigma} \otimes (x_0, \dots, x_n) \\ &= (\varphi^{n-1}[\sigma|_{J_0}]_a^{p_0, \dots, p_n} \oplus \dots \oplus \varphi^{n-1}[\sigma|_{J_t}]_a^{p_0, \dots, p_n}) \otimes (x_0, \dots, x_n) \\ &= (\varphi^{n-1}[\sigma|_{J_0}]_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n)) \oplus \dots \oplus (\varphi^{n-1}[\sigma|_{J_t}]_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n)) \end{aligned}$$

using Lemma 3.4. Here we used the notation of Definition 3.11, with the partition into intervals $[\mathbf{n}] = J_0 \sqcup \dots \sqcup J_t$. Hence, it suffices to check the claim for each such interval J , itself partitioned into intervals $J = K_0 \sqcup \dots \sqcup K_d$. Here the dimension d satisfies $d < n$, since C_σ is a face of the boundary ∂I^n . By definition of the obstruction map $\mathcal{O}(\varphi^{n-1})$, we have

$$\begin{aligned} & \varphi^{n-1}[\sigma|_J]_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n) \\ &= \varphi_a^{p_{K_0}, \dots, p_{K_d}} \otimes (x_0, \dots, x_n) \\ &= \varphi_a^{\pi_0, \dots, \pi_d} \otimes (p_{K_0}, \dots, p_{K_d}) \otimes (x_0, \dots, x_n) \end{aligned}$$

by universality of φ^d , where we denoted the projection maps $\pi_i: A^{d+1} \rightarrow A$. Since the composite

$$X \xrightarrow{(x_0, \dots, x_n)} A^{n+1} \xrightarrow{(p_{K_0}, \dots, p_{K_d})} A^{d+1}$$

is equal to $(x_{K_0}, \dots, x_{K_d}): X \rightarrow A^{d+1}$, we obtain the further simplifications:

$$\begin{aligned} &= \varphi_a^{\pi_0, \dots, \pi_d} \otimes ((p_{K_0}, \dots, p_{K_d})(x_0, \dots, x_n)) \\ &= \varphi_a^{\pi_0, \dots, \pi_d} \otimes (x_{K_0}, \dots, x_{K_d}) \\ &= \varphi_a^{x_{K_0}, \dots, x_{K_d}} \quad \text{by universality of } \varphi^d \\ &= \varphi^{n-1}[\sigma|_J]_a^{x_0, \dots, x_n}. \end{aligned} \quad \square$$

Corollary 4.5. *Let φ^{n-1} be an $(n-1)$ -distributor satisfying the universality condition 4.1 (1), and let*

$$\varphi_a^{p_0, \dots, p_n}: I^n \rightarrow \mathcal{T}(A^{n+1}, B)$$

be an extension of the obstruction map $\mathcal{O}(\varphi^{n-1})_a^{p_0, \dots, p_n}: \partial I^n \rightarrow \mathcal{T}(A^{n+1}, B)$, for each map $a \in \mathcal{T}$, depending continuously on a . Then the formula

$$\varphi_a^{x_0, \dots, x_n} := \varphi_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n)$$

defines an n -distributor φ^n based on φ^{n-1} . Note that φ^n also satisfies universality, by construction.

Proof. The formula is well-defined and continuous in its inputs a, x_0, \dots, x_n . The restriction of φ^n to the boundary ∂I^n is:

$$\begin{aligned} \varphi_a^{x_0, \dots, x_n}|_{\partial I^n} &= (\varphi_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n))|_{\partial I^n} \\ &= (\varphi_a^{p_0, \dots, p_n}|_{\partial I^n}) \otimes (x_0, \dots, x_n) \\ &= (\mathcal{O}(\varphi^{n-1})_a^{p_0, \dots, p_n}) \otimes (x_0, \dots, x_n) \quad \text{by assumption} \\ &= \mathcal{O}(\varphi^{n-1})_a^{x_0, \dots, x_n} \end{aligned}$$

by Lemma 4.4, using the fact that φ^{n-1} satisfies universality. \square

Lemma 4.6. *Let φ^{n-1} be an $(n-1)$ -distributor for \mathcal{T} satisfying the wedge condition, i.e., Definition 4.1 (2).*

- (1) *The obstruction to n -distributivity $\mathcal{O}(\varphi^{n-1}): \partial I^n \rightarrow \mathcal{T}$ satisfies the following analogous condition: For all maps $a: A \rightarrow B$ and $x: X \rightarrow A$ in \mathcal{T} , and index $0 \leq k \leq n$, the map*

$$\mathcal{O}(\varphi^{n-1})_a^{0,\dots,x,\dots,0}: \partial I^n \rightarrow \mathcal{T}(X, B)$$

is constant with value ax .

- (2) *If moreover φ^{n-1} satisfies universality, i.e., Definition 4.1 (1), then the property of $\mathcal{O}(\varphi^{n-1})$ described in the previous part is equivalent to the following: For every map $a: A \rightarrow B$ in \mathcal{T} , the composite*

$$\partial I^n \xrightarrow{\mathcal{O}(\varphi^{n-1})_a^{p_0,\dots,p_n}} \mathcal{T}(A^{n+1}, B) \xrightarrow{(i_0^*, \dots, i_n^*)} \mathcal{T}(A, B)^{n+1}$$

is constant with value (a, \dots, a) .

Proof. (1) It suffices to show that for every face $C_\sigma \subset \partial I^n$, the restriction $\mathcal{O}(\varphi^{n-1})_a^{0,\dots,x,\dots,0}|_{C_\sigma} = \varphi^{n-1}[\sigma]_a^{0,\dots,x,\dots,0}$ is constant with value ax . Consider the partition $[\mathbf{n}] = J_0 \sqcup \dots \sqcup J_t$ as in Definition 3.11, with the chosen index k satisfying $k \in J_l$ for some unique $0 \leq l \leq t$. For $i \neq l$, we have

$$\varphi^{n-1}[\sigma|_{J_i}]_a^{0,\dots,x,\dots,0} = \varphi_a^{0,\dots,0} = 0: I^{\dim \sigma|_{J_i}} \rightarrow \mathcal{T}(X, B).$$

For $i = l$, write the partition

$$J_l = K_0 \sqcup K_1 \sqcup \dots \sqcup K_d$$

as in Definition 3.11, with $k \in K_m$. Then we have

$$x_{K_j} = \begin{cases} x & \text{if } j = m \\ 0 & \text{if } j \neq m \end{cases}$$

and therefore

$$\begin{aligned} \varphi^{n-1}[\sigma|_{J_l}]_a^{0,\dots,x,\dots,0} &= \varphi_a^{x_{K_0}, \dots, x_{K_m}, \dots, x_{K_d}} \\ &= \varphi_a^{0,\dots,x,\dots,0} \\ &= ax: I^{\dim \sigma|_{J_l}} \rightarrow \mathcal{T}(X, B) \end{aligned}$$

since φ^d satisfies the wedge condition. Finally, we obtain:

$$\begin{aligned} \varphi^{n-1}[\sigma]_a^{0,\dots,x,\dots,0} &= \varphi^{n-1}[\sigma|_{J_0}]_a^{0,\dots,x,\dots,0} \oplus \dots \oplus \varphi^{n-1}[\sigma|_{J_l}]_a^{0,\dots,x,\dots,0} \oplus \dots \oplus \varphi^{n-1}[\sigma|_{J_t}]_a^{0,\dots,x,\dots,0} \\ &= 0 \oplus \dots \oplus ax \oplus \dots \oplus 0 \\ &= ax: I^{\dim \sigma} \rightarrow \mathcal{T}(X, B). \end{aligned}$$

- (2) This uses the same argument as in Lemma 4.3. □

Lemma 4.7. *Let $i: X \rightarrow Y$ be a Serre cofibration between spaces, and let L be a Serre cofibrant space. Then the map $i \times L: X \times L \rightarrow Y \times L$ is a Serre cofibration.*

The statement also holds with every instance of “Serre” replaced by “mixed”, or every instance replaced by “Hurewicz”.

Proof. More generally, if $i: X \rightarrow Y$ and $j: K \rightarrow L$ are Serre cofibrations, then their pushout-product

$$(X \times L) \cup_{X \times K} (Y \times K) \rightarrow Y \times L$$

is a Serre cofibration [18, Proposition 4.2.11] [29, Theorem 17.2.2]. The mixed model structure on **Top** also satisfies this pushout-product axiom with respect to the Cartesian product [29, Theorem 17.4.2], as does the Hurewicz model structure [29, Theorem 17.1.1]. \square

Proposition 4.8. *Let \mathcal{T} be a weakly bilinear mapping theory in which all mapping spaces $\mathcal{T}(A, B)$ are Serre cofibrant. Let $n \geq 1$, and let φ^{n-1} be a good $(n-1)$ -distributor for \mathcal{T} . Then there exists a good n -distributor φ^n for \mathcal{T} based on φ^{n-1} . Moreover, such a φ^n is unique up to homotopy rel ∂I^n .*

Proof. By Equation (4.1), it suffices to consider the universal case $x_i = p_i: A^{n+1} \rightarrow A$. Recall that the restriction $\varphi_a^{p_0, \dots, p_n}|_{\partial I^n}$ must be the obstruction map $\mathcal{O}(\varphi^{n-1}): \partial I^n \rightarrow \mathcal{T}(A^{n+1}, B)$, which is determined by φ^{n-1} . Since φ^{n-1} satisfies the wedge condition, the following square commutes:

$$\begin{array}{ccc} \partial I^n \times \mathcal{T}(A, B) & \xrightarrow{\mathcal{O}(\varphi^{n-1})} & \mathcal{T}(A^{n+1}, B) \\ \downarrow & \nearrow \varphi_a^{p_0, \dots, p_n} & \downarrow (i_0^*, \dots, i_n^*) \\ I^n \times \mathcal{T}(A, B) & \xrightarrow{\Delta \circ \text{proj}_2} & \mathcal{T}(A, B)^{n+1} \end{array}$$

$$(t, a) \longmapsto (a, \dots, a),$$

using Lemma 4.6. Since the space $\mathcal{T}(A, B)$ is Serre cofibrant by assumption, the downward map $\partial I^n \times \mathcal{T}(A, B) \rightarrow I^n \times \mathcal{T}(A, B)$ is a Serre cofibration, by Lemma 4.7. Since (i_0^*, \dots, i_n^*) is a trivial Serre fibration, there exists a dotted filler in the diagram. The top triangle guarantees that the collection of n -cubes

$$\varphi^n := \{\varphi_a^{p_0, \dots, p_n} \otimes (x_0, \dots, x_n) \mid a, x_0, \dots, x_n \in \mathcal{T}\}$$

defines an n -distributor for \mathcal{T} which is based on φ^{n-1} , using Corollary 4.5. By construction, φ^n satisfies universality. The bottom triangle guarantees that φ^n also satisfies the wedge condition, hence is good.

For uniqueness, let φ and φ' be two good extensions of $\mathcal{O}(\varphi^{n-1})$ to I^n . These jointly define a map

$$\begin{array}{ccc} (I^n \times \mathcal{T}(A, B)) \cup_{\partial I^n \times \mathcal{T}(A, B)} (I^n \times \mathcal{T}(A, B)) & \xrightarrow{\varphi \cup \varphi'} & \mathcal{T}(A^{n+1}, B) \\ \cong \downarrow & \nearrow & \\ (I^n \cup_{\partial I^n} I^n) \times \mathcal{T}(A, B) & & \\ \cong \downarrow & \nearrow & \\ S^n \times \mathcal{T}(A, B). & & \end{array}$$

Here, we used the fact that the functor $- \times Z: \mathbf{Top} \rightarrow \mathbf{Top}$ preserves colimits. Again, there exists a filler in the diagram

$$\begin{array}{ccc} S^n \times \mathcal{T}(A, B) & \xrightarrow{\varphi \cup \varphi'} & \mathcal{T}(A^{n+1}, B) \\ \downarrow & \nearrow & \downarrow (i_0^*, \dots, i_n^*) \\ D^n \times \mathcal{T}(A, B) & \xrightarrow{\Delta \circ \text{proj}_2} & \mathcal{T}(A, B)^{n+1}, \end{array}$$

which provides a homotopy rel ∂I^n between φ and φ' . \square

Proof of Theorem 3.8. Starting from the 0-distributor φ^0 , inductively choose a good n -distributor φ^n based on φ^{n-1} , for all $n \geq 1$. \square

5. THE KRISTENSEN DERIVATION

In this section, we fix the prime $p = 2$ and work with the mod 2 Eilenberg–MacLane mapping theory \mathcal{EM} , as in Definition 2.9. Recall that $K_n = \text{sh}^n K_0$ denotes our preferred model for $\Sigma^n H\mathbb{F}_2$.

5.1. The Kristensen derivation from 1-distributivity. Let φ^1 be a good 1-distributor for \mathcal{EM} ; recall that φ^1 consists of a collection of paths $\varphi_a^{x,y}$ of the form illustrated here:

$$a(x+y) \xrightarrow{\varphi_a^{x,y}} ax+ay.$$

For $n \leq 2$, we usually denote the inputs x_0, x_1, x_2 by x, y, z . The following terminology and notation follows [5, §4.2].

Definition 5.1. The **linearity tracks** for \mathcal{EM} are the homotopy classes of the paths $\varphi_a^{x,y}$ rel ∂I , i.e., the tracks

$$\Gamma_a^{x,y} := \{\varphi_a^{x,y}\}.$$

Note that by Proposition 4.8, $\Gamma_a^{x,y}$ is well-defined, i.e., independent of the choice of a good 1-distributor φ^1 . Now take an element of the Steenrod algebra $a \in \mathcal{A}_m$ of degree m , represented by a map $a: K_n \rightarrow K_{n+m}$. For our purposes, we may as well choose $n = 0$, since the shift functor induces a weak equivalence $\text{sh}^n: \mathcal{EM}(K_0, K_m) \xrightarrow{\sim} \mathcal{EM}(K_n, K_{n+m})$. Taking $x = y = 1_{K_0}$, the linearity track $\Gamma_a^{1,1}$ is a track in $\mathcal{EM}(K_0, K_m)$ of the form

$$0 = a0 = a(1+1) \xrightarrow{\Gamma_a^{1,1}} a1 + a1 = a + a = 0.$$

Here, we used the fact that K_n is an \mathbb{F}_2 -vector space object, by Lemma A.8. The track $\Gamma_a^{1,1}$ is a well defined class

$$\begin{aligned} \kappa(a) &:= \Gamma_a^{1,1} \in \pi_1 \mathcal{EM}(K_0, K_m) = \pi_1 \mathbf{Spec}(K_0, K_m) \\ &= \pi_0 \Omega \mathbf{Spec}(K_0, K_m) \\ &= \pi_0 \mathbf{Spec}(K_0, \Omega K_m) \\ &= [K_0, \Omega K_m] \\ &= [H\mathbb{F}_2, \Sigma^{m-1} H\mathbb{F}_2] \\ &= \mathcal{A}_{m-1}. \end{aligned}$$

This defines a function $\kappa: \mathcal{A}_m \rightarrow \mathcal{A}_{m-1}$, or in other words, a function $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ of degree -1 .

In what follows, we will use the linearity track equations [5, Theorem 4.2.5], i.e., certain equations satisfied by the 1-tracks $\Gamma_a^{x,y}$. More precisely, we will need the following three equations.

Lemma 5.2. *The linearity 1-tracks $\Gamma_a^{x,y}$ satisfy the following equations.*

- (1) *Precomposition:* $\Gamma_a^{xd,yd} = \Gamma_a^{x,y} d$.
- (2) *Left linearity:* $\Gamma_{a+a'}^{x,y} = \Gamma_a^{x,y} + \Gamma_{a'}^{x,y}$.
- (3) *Product rule:* $\Gamma_{ba}^{x,y} = \Gamma_b^{ax,ay} \square b\Gamma_a^{x,y}$.

Lemma 5.3. *The function $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ is linear, i.e., preserves addition.*

Proof. This follows from linearity of $\Gamma_a^{x,y}$ in the input a :

$$\kappa(a + a') = \Gamma_{a+a'}^{1,1} = \Gamma_a^{1,1} + \Gamma_{a'}^{1,1} = \kappa(a) + \kappa(a').$$

□

Lemma 5.4. *The function $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ is a derivation.*

Proof. This is stated and sketched in [5, Lemma 4.5.5]. Using the precomposition equation and product rule from Lemma 5.2, we obtain:

$$\begin{aligned} \kappa(ba) &= \Gamma_{ba}^{1,1} \\ &= \Gamma_b^{a,a} \square b \Gamma_a^{1,1} \\ &= \Gamma_b^{1,1} a \square b \Gamma_a^{1,1} \\ &= \kappa(b)a + b\kappa(a). \end{aligned}$$

We used the fact that via the identification $\pi_1 \mathcal{EM}(K_0, K_m) \cong \mathcal{A}_{m-1}$, concatenation of loops corresponds to addition in \mathcal{A}_{m-1} , while the \otimes -product of a loop with a 0-cube corresponds to multiplication in \mathcal{A} . □

Proposition 5.5. *Applied to Steenrod squares, the function κ satisfies*

$$\kappa(\text{Sq}^m) = \text{Sq}^{m-1}.$$

In particular, κ agrees with the Kristensen derivation [22, §2].

Proof. This is proved in [5, Theorem 4.5.8], using work in [22]. □

The existence of the derivation $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ is a non-trivial property of the Steenrod algebra, which can be checked explicitly using the Adem relations; c.f. [22, §2].

Remark 5.6. It was pointed out to us by Fernando Muro that the Kristensen derivation $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ is also obtained from the construction described in [7, §1]. More precisely, consider the ring spectrum $R = \text{End}_S(H\mathbb{F}_p, H\mathbb{F}_p)$, the endomorphism ring spectrum of $H\mathbb{F}_p$ as a module over the sphere spectrum S . The homotopy groups of R are the Steenrod algebra with reversed grading:

$$\pi_m R = [S^m \wedge H\mathbb{F}_p, H\mathbb{F}_p] \cong [H\mathbb{F}_p, S^{-m} \wedge H\mathbb{F}_p] = \mathcal{A}_{-m}.$$

The unit map $\eta: S \rightarrow R$ induces on homotopy the map

$$\mathbb{Z} \cong \pi_0 S \xrightarrow{\pi_0 \eta} \pi_0 R \cong \mathbb{F}_p$$

which sends the class of the identity to the class of the identity. Taking the class $p \in \pi_0 S$ which lies in the kernel of $\pi_0 \eta$, the construction in [7, §1] yields a function

$$\theta(p): \pi_m R \rightarrow \pi_{m+1} R$$

which in this case is independent of the choice of nullhomotopy of $p1_{H\mathbb{F}_p}$, because the indeterminacy lives in $\pi_{0+1} R = \mathcal{A}_{-1} = 0$. The function $\theta(p)$ sends a class $a \in \pi_m R$ to a certain self-track of zero $ap \rightarrow pa$. In the case $p = 2$, this track coincides with the linearity track $\kappa(a) = \{\varphi_a^{1,1}\}: a2 = a(1+1) \rightarrow (1+1)a = 2a$.

5.2. Linearity 2-tracks.

Definition 5.7. Let \mathcal{T} be a weakly bilinear mapping theory with Serre cofibrant mapping spaces. Let φ^2 be a good 2-distributor for \mathcal{T} . We call the homotopy class $\text{rel } \partial I^2$ of $\varphi_a^{x,y,z}: I^2 \rightarrow \mathcal{T}$, denoted $\{\varphi_a^{x,y,z}\}$, a **linearity 2-track**.

Again by Proposition 4.8, $\{\varphi_a^{x,y,z}\}$ is determined by the underlying 1-distributor φ^1 . In this subsection, we work out a few equations satisfied by the linearity 2-tracks, analogous to the equations satisfied by the linearity 1-tracks $\{\varphi_a^{x,y}\}$ listed in Lemma 5.2.

Lemma 5.8 (Precomposition). *For all inputs $a, x, y, z, d \in \mathcal{T}$, the following equation of 2-tracks holds: $\{\varphi_a^{xd,yd,zd}\} = \{\varphi_a^{x,y,z}\}d$.*

Proof. Since φ^2 satisfies the universality condition, the equality holds even at the level of 2-cubes, not merely 2-tracks:

$$\begin{aligned} \varphi_a^{xd,yd,zd} &= \varphi_a^{p_0,p_1,p_2} \otimes (xd, yd, zd) \\ &= \varphi_a^{p_0,p_1,p_2} \otimes (x, y, z) \otimes d \\ &= \varphi_a^{x,y,z} \otimes d. \end{aligned}$$

□

Next, we work out the analogue of the left linearity equation of 1-tracks $\Gamma_{a+a'}^{x,y} = \Gamma_a^{x,y} + \Gamma_{a'}^{x,y}$. Note that $\{\varphi_{a+a'}^{x,y,z}\}$ and $\{\varphi_a^{x,y,z}\} + \{\varphi_{a'}^{x,y,z}\}$ are usually different, since they do not agree on the boundary ∂I^2 . To compare them, we need some correction 2-tracks.

Notation 5.9. Consider maps $a, a': A \rightarrow B$ in \mathcal{T} . By the argument in Proposition 4.8, there exists a path homotopy

$$L_{a,a'}^{p_0,p_1}: \varphi_{a+a'}^{p_0,p_1} \Rightarrow \varphi_a^{p_0,p_1} + \varphi_{a'}^{p_0,p_1}$$

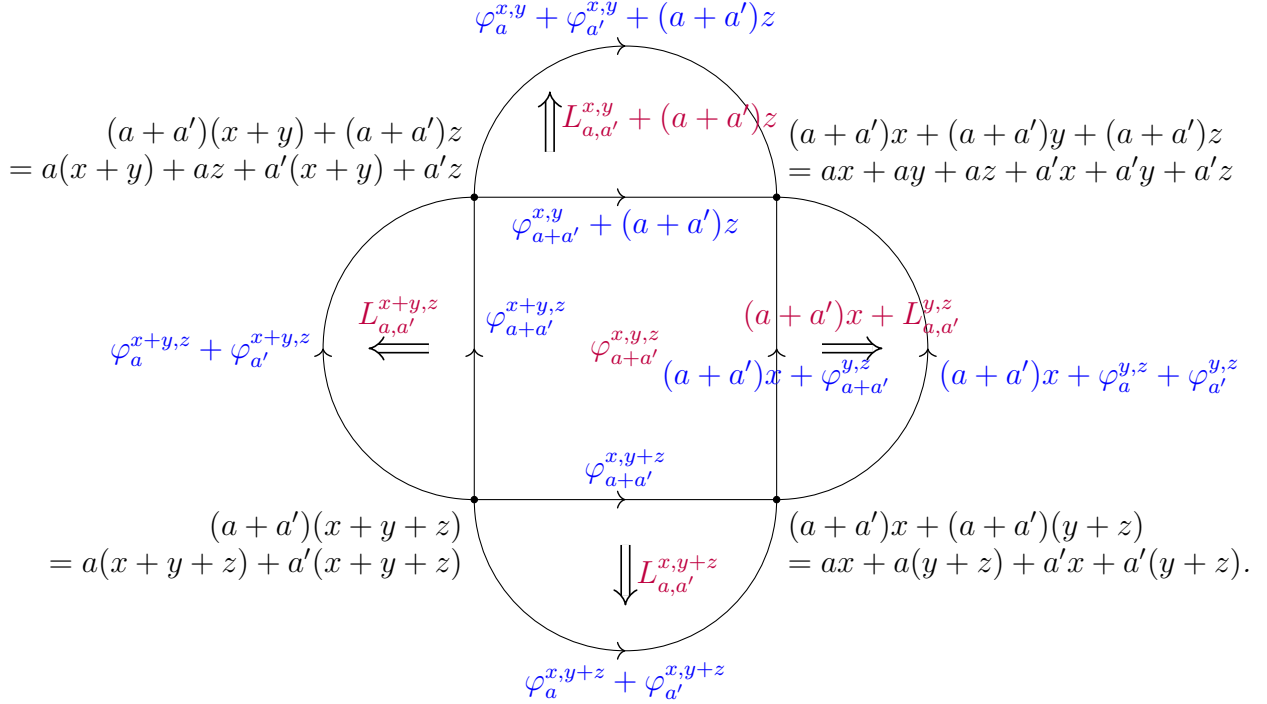
between paths in $\mathcal{T}(A \times A, B)$ such that for all k , the restriction $i_k^* L_{a,a'}^{p_0,p_1}$ is the constant 2-cube at $a + a' \in \mathcal{T}(A, B)$. Moreover, such a path homotopy is unique up to homotopy $\text{rel } \partial I^2$, i.e., yields a well-defined globular 2-track $\{L_{a,a'}^{p_0,p_1}\}$. For arbitrary maps $x, y: X \rightarrow A$ in \mathcal{T} , define

$$L_{a,a'}^{x,y} := L_{a,a'}^{p_0,p_1} \otimes (x, y),$$

which is a path homotopy in $\mathcal{T}(X, B)$ as illustrated here:

$$\begin{array}{ccc} a(x+y) + a'(x+y) & \xrightarrow{\varphi_a^{x,y} + \varphi_{a'}^{x,y}} & ax + ay + a'x + a'y \\ \parallel & \uparrow L_{a,a'}^{x,y} & \parallel \\ (a+a')(x+y) & \xrightarrow{\varphi_{a+a'}^{x,y}} & (a+a')x + (a+a')y \end{array}$$

Lemma 5.10 (Left linearity). *The 2-track illustrated in Figure 5.1 is equal to $\{\varphi_a^{x,y,z}\} + \{\varphi_{a'}^{x,y,z}\}$.*

FIGURE 5.1. Relating the 2-tracks $\{\varphi_a^{x,y,z}\}$ and $\{\varphi_a^{x,y,z}\} + \{\varphi_{a'}^{x,y,z}\}$.

Proof. The illustrated 2-track and $\{\varphi_a^{x,y,z}\} + \{\varphi_{a'}^{x,y,z}\}$ have the same restriction to the boundary ∂I^2 , namely $\mathcal{O}(\varphi^1)_a^{x,y,z} + \mathcal{O}(\varphi^1)_{a'}^{x,y,z}$. When all inputs x, y, z are zero except one x_k , then the illustrated 2-track and $\{\varphi_a^{x,y,z}\} + \{\varphi_{a'}^{x,y,z}\}$ are both the constant 2-track at $ax_k + a'x_k = (a+a')x_k \in \mathcal{T}(X, B)$. By universality, it suffices to prove the claim in the case $x_i = p_i: A^3 \rightarrow A$. The claimed equality of 2-tracks then follows from the uniqueness argument in Proposition 4.8. \square

Next, we turn to the product rule. The linearity 1-tracks $\Gamma_a^{x,y}$ satisfy the equation

$$\Gamma_{ba}^{x,y} = \Gamma_b^{ax,ay} \square b\Gamma_a^{x,y}.$$

As before, let us exhibit a canonical globular 2-track that witnesses this equality of 1-tracks.

Notation 5.11. Consider maps $a: A \rightarrow B$ and $b: B \rightarrow C$ in \mathcal{T} . Denote by

$$P_{b,a}^{x,y}: \varphi_b^{ax,ay} \square b\varphi_a^{x,y} \Rightarrow \varphi_{ba}^{x,y}$$

the path homotopy in $\mathcal{T}(X, C)$ as illustrated here:

$$\begin{array}{ccccc} & & \varphi_{ba}^{x,y} & & \\ & & \uparrow P_{b,a}^{x,y} & & \\ ba(x+y) & \xrightarrow{b\varphi_a^{x,y}} & b(ax+ay) & \xrightarrow{\varphi_b^{ax,ay}} & bax+bay \end{array}$$

defined similarly to Notation 5.9, yielding a well-defined globular 2-track $\{P_{b,a}^{x,y}\}$.

Distributors can be generalized by letting the inputs $x_i \in \mathcal{T}$ be continuous families instead of points, and then applying the distributor pointwise. We make this precise in the following notation.

Notation 5.12. Let $a: A \rightarrow B$ be a map in \mathcal{T} , and $v: V \rightarrow \mathcal{T}(X, A)$ and $w: W \rightarrow \mathcal{T}(X, A)$ maps of spaces. As in Definition 3.3, the external addition $v \oplus w: V \times W \rightarrow \mathcal{T}(X, A)$ is the composite

$$V \times W \xrightarrow{v \times w} \mathcal{T}(X, A) \times \mathcal{T}(X, A) \xrightarrow{+} \mathcal{T}(X, A).$$

The 1-distributor φ_a^1 applied to the inputs v and w is the map $\varphi_a^{v,w}: V \times W \times I \rightarrow \mathcal{T}(X, B)$ defined as the composite

$$V \times W \times I \xrightarrow{v \times w \times \text{id}} \mathcal{T}(X, A) \times \mathcal{T}(X, A) \times I \xrightarrow{\varphi_a^1} \mathcal{T}(X, B),$$

viewed as a homotopy from $a(v \oplus w)$ to $av \oplus aw$.

Lemma 5.13 (Product rule). *The 2-track illustrated in Figure 5.2 is equal to $\{\varphi_{ba}^{x,y,z}\}$.*

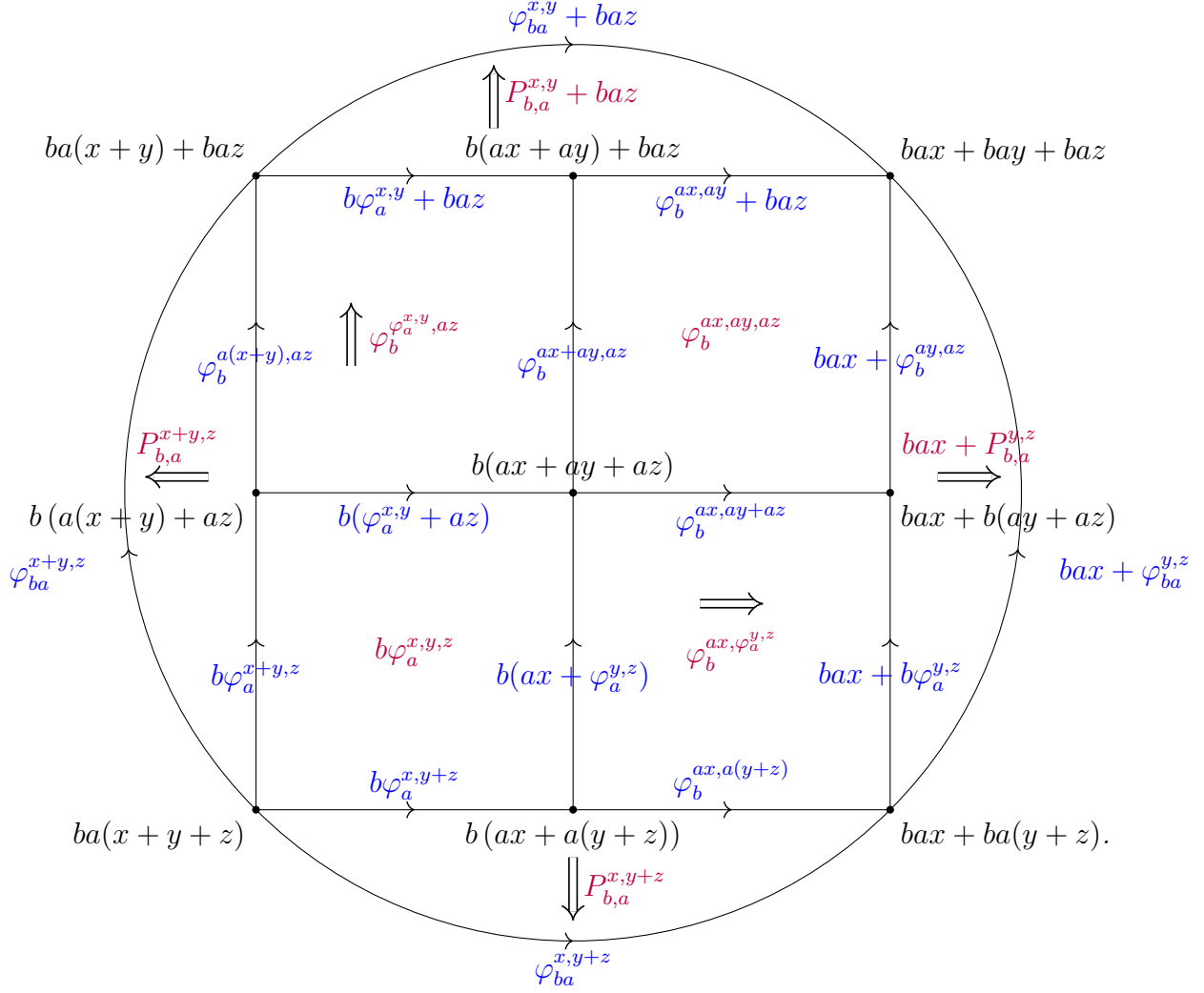


FIGURE 5.2. Relating the 2-track $\{\varphi_{ba}^{x,y,z}\}$ to $\{\varphi_a^{x,y,z}\}$ and $\{\varphi_b^{ax,ay,az}\}$.

Proof. This is similar to the proof of Lemma 5.10. □

Lemma 5.14. *Consider a map $a: A \rightarrow B$, a 2-cube $u: I^2 \rightarrow \mathcal{T}(X, A)$, a point $y \in \mathcal{T}(X, A)$. Then the 2-track illustrated in Figure 5.3 is equal to $\{au + ay\}$.*

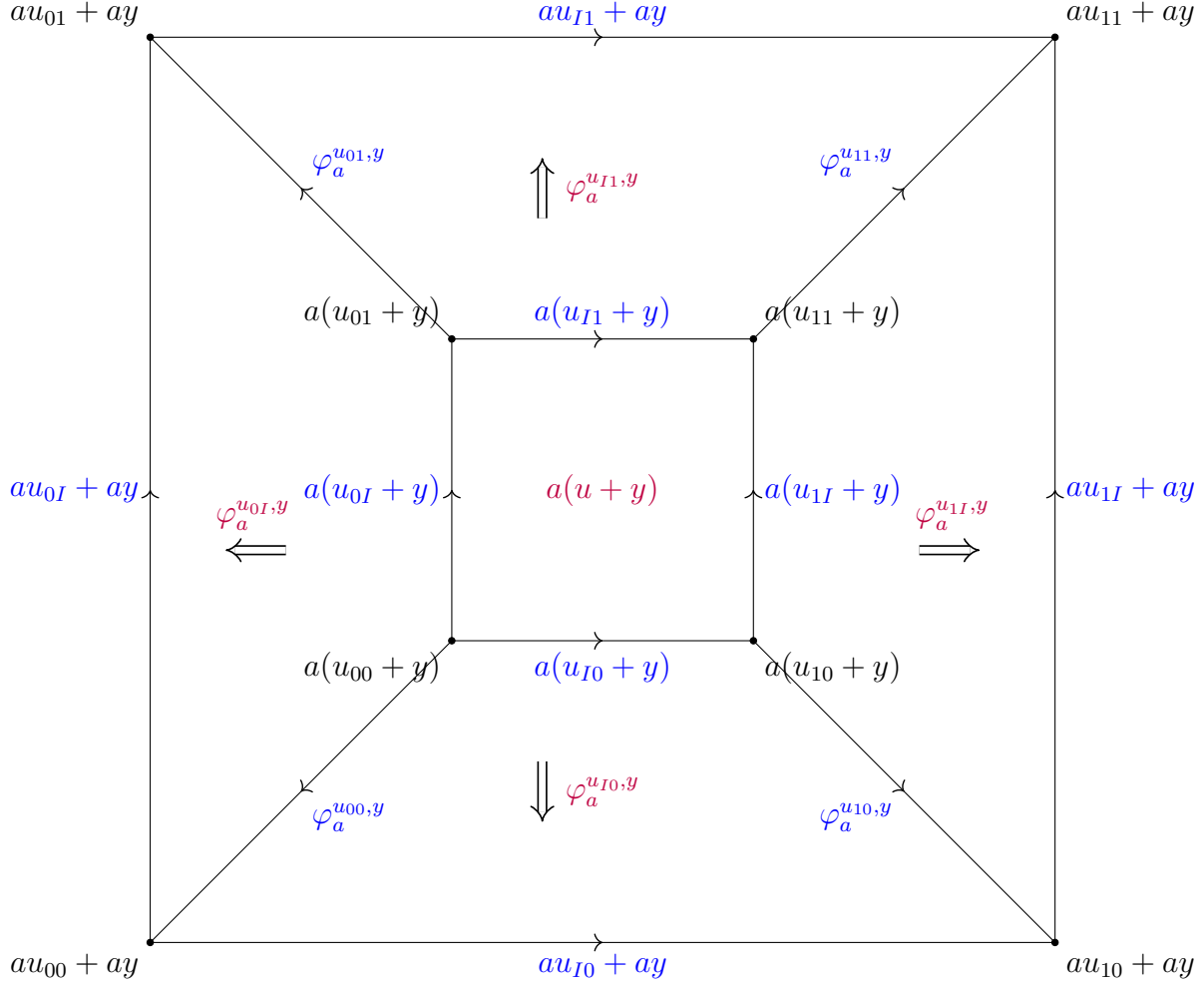
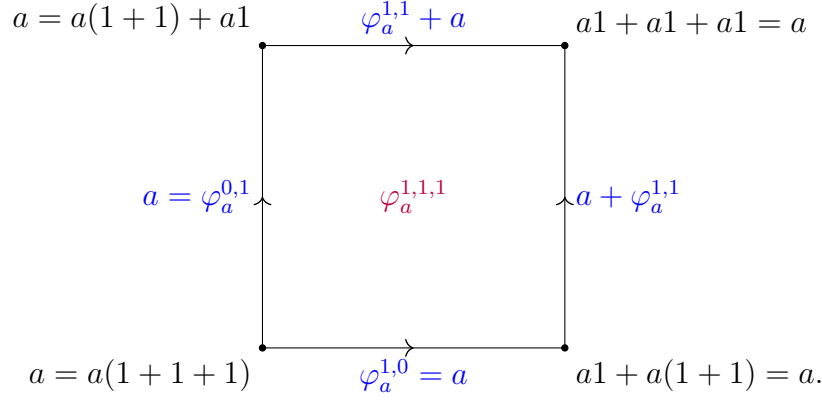


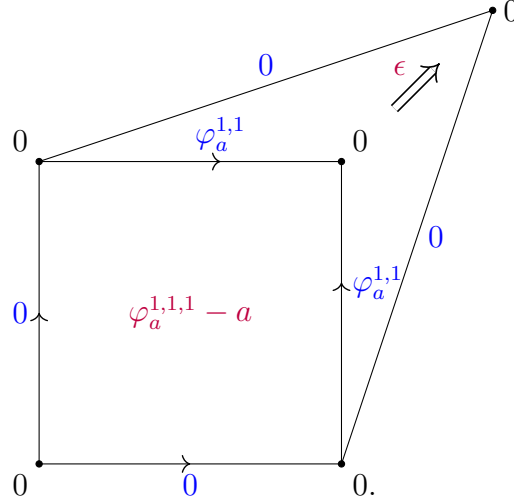
FIGURE 5.3. Comparing the 2-tracks $\{a(u + y)\}$ and $\{au + ay\}$.

Proof. The 3-cube $\varphi_a^{u,y}: I^3 \rightarrow \mathcal{T}(X, A)$ provides a homotopy rel ∂I^2 between the illustrated 2-track and $\{au + ay\}$. \square

5.3. Two-dimensional analogue of the derivation. Let φ^2 be a good 2-distributor for \mathcal{EM} . As in the section 5.1, start with an element of the Steenrod algebra $a \in \mathcal{A}_m$, represented by a map $a: K_0 \rightarrow K_m$. The 2-cube $\varphi_a^{1,1,1}: I^2 \rightarrow \mathcal{EM}$ restricts to the boundary ∂I^2 as illustrated in Figure 5.4.

FIGURE 5.4. The 2-cube $\varphi_a^{1,1,1}: I^2 \rightarrow \mathcal{EM}$.

Note that the equations $\varphi_a^{1,0} = \text{cst}_a = \varphi_a^{0,1}$ are instances of the wedge condition satisfied by the 1-distributor φ^1 . Subtracting a pointwise yields the 2-cube $\varphi_a^{1,1,1} - a: I^2 \rightarrow \mathcal{EM}$ illustrated in Figure 5.5. The top right part uses the canonical path homotopy $\epsilon: \gamma^\square \square \gamma \Rightarrow \text{cst}_{\gamma(0)}$ to the constant path at $\gamma(0)$.

FIGURE 5.5. The 2-cube $\varphi_a^{1,1,1} - a: I^2 \rightarrow \mathcal{EM}$ and a correction term.

Taking the homotopy class $\text{rel } \partial I^2$, this construction yields a well-defined class

$$\lambda(a) \in \pi_2 \mathcal{EM}(K_0, K_m) \cong \mathcal{A}_{m-2},$$

and thus a function $\lambda: \mathcal{A} \rightarrow \mathcal{A}$ of degree -2 .

Lemma 5.15. *The function $\lambda: \mathcal{A} \rightarrow \mathcal{A}$ is linear, i.e., preserves addition.*

Proof. Let $a, a' \in \mathcal{A}_m$. Applying Lemma 5.10 to the 2-track $\{\varphi_{a+a'}^{1,1,1}\}$ and using the fact that $\{L_{a,a'}^{0,1}\}$ and $\{L_{a,a'}^{1,0}\}$ are both the constant 2-track at $a + a'$, we obtain $\lambda(a + a') = \lambda(a) + \lambda(a')$. \square

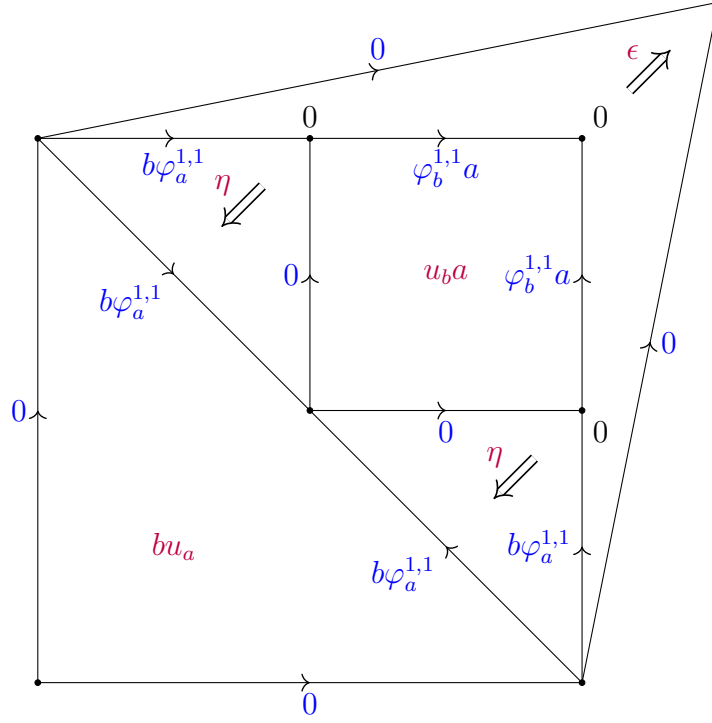
Proposition 5.16. *The function $\lambda: \mathcal{A} \rightarrow \mathcal{A}$ is a derivation.*

The diagram consists of 9 nodes arranged in a 3x3 grid. The nodes are connected by arrows representing various algebraic operations:

- Top Row:**
 - Left node: $\varphi_{ba}^{1,1} + ba$ (above)
 - Middle node: ba (above)
 - Right node: ba (above)
- Middle Row:**
 - Left node: $b\varphi_a^{1,1} + ba$ (above)
 - Middle node: ba (above)
 - Right node: $ba + \varphi_b^{1,1}a$ (above)
- Bottom Row:**
 - Left node: $b(\varphi_a^{1,1} + a)$ (above)
 - Middle node: ba (above)
 - Right node: ba (above)
- Left Column:**
 - Top node: ba (left)
 - Middle node: ba (left)
 - Bottom node: ba (left)
- Right Column:**
 - Top node: $ba + \varphi_b^{1,1}a$ (left)
 - Middle node: $ba + P_{b,a}^{1,1}$ (left)
 - Bottom node: $ba + b\varphi_a^{1,1}$ (left)
- Bottom Row (continued):**
 - Left node: $b\varphi_a^{1,1,1}$ (below)
 - Middle node: $b(a + \varphi_a^{1,1})$ (below)
 - Right node: $\varphi_b^{a, \varphi_a^{1,1}}$ (below)
- Right Side:**
 - A large curved arrow on the right side, labeled $ba + \varphi_{ba}^{1,1}$ (right).

FIGURE 5.6. The 2-track $\{\varphi_{ba}^{1,1,1}\}$.

We used the fact that both $P_{b,a}^{1,0}$ and $P_{b,a}^{0,1}$ are the constant 2-track at $ba \in \mathcal{EM}(A, C)$. We also used the precomposition equation from Lemma 5.8. Denote the 2-track $u_a := \{\varphi_a^{1,1} - a\}$. Applying Lemma 5.14 to $b\varphi_a^{1,1} = b(u_a + a)$, subtracting ba pointwise everywhere, and applying the correction 2-track ϵ in the upper right part of the diagram, we deduce that the 2-track $\lambda(ba)$ is given as in Figure 5.7.

FIGURE 5.7. The 2-track $\lambda(ba)$.

There, we denote by $\eta: \text{cst}_{\gamma(1)} \square \gamma \Rightarrow \gamma$ and $\eta: \gamma \square \text{cst}_{\gamma(0)} \Rightarrow \gamma$ the canonical path homotopies, whose 2-tracks are well-defined. Straightforward manipulations of 2-tracks yield the claimed equality $\lambda(ba) = \lambda(b)a + b\lambda(a)$. \square

When working with mod 2 coefficients, a composite of derivations is still a derivation. In particular, we have:

$$\begin{aligned} \kappa^2(ab) &= \kappa^2(a)b + \kappa(a)\kappa(b) + \kappa(a)\kappa(b) + a\kappa^2(b) \\ &= \kappa^2(a)b + a\kappa^2(b). \end{aligned}$$

We leave the following question to the reader.

Question 5.17. Is the derivation $\lambda: \mathcal{A} \rightarrow \mathcal{A}$ given by the composite $\lambda = \kappa^2$?

6. HOMOTOPY INVARIANCE

In this section, we study to what extent an n -distributor is a homotopy invariant structure. This section is mostly independent of Sections 4 and 5. Unlike in those two sections, goodness of distributors will play no role. Also, finite products in \mathcal{T} , which were crucial to the construction of distributors, are not used here. Hence, instead of left linear mapping theories, we work with left linear **Top**_{*}-enriched categories.

6.1. Pulling back distributors.

Lemma 6.1. *Let \mathcal{S} and \mathcal{T} be **Top**-enriched categories in which all mapping spaces are topological abelian groups. Let $F: \mathcal{S} \rightarrow \mathcal{T}$ be a **Top**-enriched functor such that for all objects A, B of \mathcal{S} , the induced map*

$$F: \mathcal{S}(A, B) \rightarrow \mathcal{T}(FA, FB)$$

is a group homomorphism. Then F preserves external addition of cubes, i.e., given $a: I^m \rightarrow \mathcal{S}(A, B)$ and $b: I^n \rightarrow \mathcal{S}(A, B)$, the equality

$$F(a \oplus b) = F(a) \oplus F(b)$$

of $(m + n)$ -cubes holds.

Proof. The diagram

$$\begin{array}{ccccc} I^{m+n} = I^m \times I^n & \xrightarrow{a \times b} & \mathcal{S}(A, B) \times \mathcal{S}(A, B) & \xrightarrow{+} & \mathcal{S}(A, B) \\ & \searrow F(a) \times F(b) & \downarrow F \times F & & \downarrow F \\ & & \mathcal{T}(FA, FB) \times \mathcal{T}(FA, FB) & \xrightarrow{+} & \mathcal{T}(FA, FB) \end{array}$$

commutes. □

Lemma 6.2. *Let $F: \mathcal{S} \rightarrow \mathcal{T}$ be a morphism of left linear \mathbf{Top}_* -enriched categories, and φ^{n-1} an $(n - 1)$ -distributor for \mathcal{S} . Then every proper subcube $C_\sigma \subset I^n$, we have*

$$F(\varphi^{n-1}[\sigma]) = (F\varphi^{n-1})[\sigma]$$

as maps $C_\sigma \rightarrow \mathcal{T}$. Consequently the obstruction map satisfies

$$F(\mathcal{O}(\varphi^{n-1})) = \mathcal{O}(F\varphi^{n-1})$$

as maps $\partial I^n \rightarrow \mathcal{T}$.

Proof. Consider the partition $\{0, 1, \dots, n\} = J_0 \sqcup J_1 \sqcup \dots \sqcup J_t$ as in Definition 3.11. Then we have

$$\begin{aligned} F(\varphi^{n-1}[\sigma]) &= F(\varphi^{n-1}[\sigma|_{J_0}] \oplus \dots \oplus \varphi^{n-1}[\sigma|_{J_t}]) \\ &= F(\varphi^{n-1}[\sigma|_{J_0}]) \oplus \dots \oplus F(\varphi^{n-1}[\sigma|_{J_t}]) && \text{by Lemma 6.1} \\ &= (F\varphi^{n-1})[\sigma|_{J_0}] \oplus \dots \oplus (F\varphi^{n-1})[\sigma|_{J_t}] && \text{using } F(x + x') = Fx + Fx' \\ &= (F\varphi^{n-1})[\sigma]. \end{aligned}$$

□

Lemma 6.3. *Let $i: A \rightarrow X$ be a mixed cofibration and $q: Y \xrightarrow{\sim} Z$ a weak equivalence. Given a commutative square as in the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \tilde{g} \nearrow & \downarrow q \\ X & \xrightarrow{g} & Z, \end{array}$$

there exists a map $\tilde{g}: X \rightarrow Y$ making the upper triangle commute strictly, and the lower triangle commute up to homotopy rel A .

Proof. Recall that the mapping path space $P(q) = Y \times_Z Z^I$ provides a (functorial) factorization of $q: Y \rightarrow Z$ into a strong deformation retract $c: Y \rightarrow P(q)$ followed by a Hurewicz fibration $p: P(q) \rightarrow Z$. Denote the retraction map by $r: P(q) \rightarrow Y$. In our case, $p: P(q) \rightarrow Z$

is also a weak equivalence, since q was. In the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & Y \\
 \downarrow i & \searrow cf & \swarrow c \\
 & P(q) & \\
 \uparrow g' & \nwarrow p & \downarrow q \\
 X & \xrightarrow{g} & Z
 \end{array}$$

(Note: In the original image, there are additional labels: a curved arrow r from Y to $P(q)$, a curved arrow \sim from $P(q)$ to Z , and a curved arrow \sim from Y to Z .)

there is a map $g': X \rightarrow P(q)$ making the two adjacent triangles commute. Indeed, $i: A \hookrightarrow X$ is a mixed cofibration, whereas $p: P(q) \twoheadrightarrow Z$ is a mixed trivial fibration (i.e., a Hurewicz fibration which is also a weak equivalence). Take $\tilde{g} = rg': X \rightarrow Y$. This map satisfies

$$\tilde{g}i = rg'i = rcf = f$$

and

$$\begin{aligned}
 q\tilde{g} &= qrg' \\
 &= pcrg' \\
 &\simeq pg' \text{ rel } A \\
 &= pg' \\
 &= g
 \end{aligned}$$

as desired. □

Proposition 6.4 (Pulling back distributors). *Let $F: \mathcal{S} \rightarrow \mathcal{T}$ be a morphism of left linear \mathbf{Top}_* -categories such that for all objects A, B of \mathcal{S} , the induced map $F: \mathcal{S}(A, B) \xrightarrow{\sim} \mathcal{T}(FA, FB)$ is a weak equivalence. (We do not assume that the functor $\pi_0 F: \pi_0 \mathcal{S} \rightarrow \pi_0 \mathcal{T}$ is essentially surjective.) Assume that every mapping space $\mathcal{S}(A, B)$ has the homotopy type of a CW complex. If \mathcal{T} is N -distributive for some $N \geq 1$ (or $N = \infty$), then \mathcal{S} is N -distributive.*

Proof. Let ψ^N be an N -distributor for \mathcal{T} . We will prove the statement by induction, using the following condition for $n \leq N$.

(Step- n)

- There is given an n -distributor φ^n for \mathcal{S} , based on φ^{n-1} .
- There is given a homotopy

$$h^n: \psi^n F \simeq F \varphi^n$$

which is compatible with the previous steps in the following sense. For every proper subcube $C_\sigma \subset I^n$, of dimension $\dim \sigma = d < n$, the restriction of h^n to C_σ satisfies

$$h^n|_{C_\sigma \times I} = h^d[\sigma]: (\psi^d F)[\sigma] \simeq F(\varphi^d[\sigma]).$$

Here $\psi^n F$ denotes the collection of cubes

$$\psi^n F = \left\{ \psi_{Fa}^{F x_0, \dots, F x_n} \mid a, x_0, \dots, x_n \in \mathcal{S} \right\}$$

and $h^d[\sigma]$ is defined by the analogue of the formula that defines $\varphi^d[\sigma]$, applied at each time of the homotopy.

Base case $n = 0$. The 0-distributor φ^0 for \mathcal{S} satisfies $F\varphi^0 = \psi^0 F$, i.e., for $a, x_0 \in \mathcal{S}$, we have

$$F(\varphi_a^{x_0}) = F(ax_0) = (Fa)(Fx_0) = \psi_{Fa}^{Fx_0}$$

using that F is a functor. Take h^0 to be the stationary homotopy between $F\varphi^0$ and $\psi^0 F$.

Inductive step from $n - 1$ to n . The two composites in the square

$$\begin{array}{ccc} \partial I^n \times \mathcal{S}(A, B) \times \mathcal{S}(X, A)^{n+1} & \xrightarrow{\mathcal{O}(\varphi^{n-1})} & \mathcal{S}(X, B) \\ \downarrow \iota & \nearrow \mathcal{O}(h^{n-1}) & \downarrow \sim F \\ I^n \times \mathcal{S}(A, B) \times \mathcal{S}(X, A)^{n+1} & \xrightarrow{\psi^n F} & \mathcal{T}(FX, FB) \\ \downarrow \text{id} \times F \times F & \nearrow \psi^n & \\ I^n \times \mathcal{T}(FA, FB) \times \mathcal{T}(FX, FA)^{n+1} & & \end{array}$$

are $\psi^n F|_{\partial I^n} = \mathcal{O}(\psi^{n-1} F)$ and $F\mathcal{O}(\psi^{n-1})$. By induction hypothesis and Lemma 6.2, the given homotopies h^{n-1} define a homotopy

$$\mathcal{O}(h^{n-1}): \mathcal{O}(\psi^{n-1} F) \simeq F\mathcal{O}(\varphi^{n-1}).$$

By Lemma 4.7, the map $\partial I^n \times \mathcal{S}(A, B) \times \mathcal{S}(X, A)^{n+1} \hookrightarrow I^n \times \mathcal{S}(A, B) \times \mathcal{S}(X, A)^{n+1}$ is a Hurewicz cofibration. By the homotopy extension property, there is a homotopy

$$\tilde{h}^n: I^n \times \mathcal{S}(A, B) \times \mathcal{S}(X, A)^{n+1} \times I \rightarrow \mathcal{T}(FX, FB)$$

extending $\mathcal{O}(h^{n-1})$ and starting at $\psi^n F$. Denote the end of the homotopy by $\tilde{\psi}^n F := \tilde{h}_1^n$, which satisfies

$$\tilde{\psi}^n F|_{\partial I^n} = \tilde{h}_1^n|_{\partial I^n} = \mathcal{O}(h^{n-1})_1 = F\mathcal{O}(\varphi^{n-1}).$$

Recall that spaces of the homotopy type of a CW complex are precisely the cofibrant objects in the mixed model structure on **Top**. In the commutative square

$$\begin{array}{ccc} \partial I^n \times \mathcal{S}(A, B) \times \mathcal{S}(X, A)^{n+1} & \xrightarrow{\mathcal{O}(\varphi^{n-1})} & \mathcal{S}(X, B) \\ \downarrow & \nearrow \varphi^n & \downarrow F \\ I^n \times \mathcal{S}(A, B) \times \mathcal{S}(X, A)^{n+1} & \xrightarrow{\tilde{\psi}^n F} & \mathcal{T}(FX, FB) \end{array}$$

$\uparrow \gamma^n$

there is a map $\varphi^n: I^n \rightarrow \mathcal{S}$ making the top triangle commute strictly and the bottom triangle commute up to homotopy rel ∂I^n , by Lemma 6.3. Thus φ^n is an n -distributor for \mathcal{S} , based on φ^{n-1} . Now, define h^n as the concatenation of the two homotopies

$$\psi^n F \xrightarrow[\tilde{h}^n]{\simeq} \tilde{\psi}^n F \xrightarrow[\gamma^n]{\simeq} F\varphi^n.$$

Since the homotopy γ^n is rel ∂I^n and the homotopy \tilde{h}^n restricts to $\mathcal{O}(h^{n-1})$ on ∂I^n , the homotopy h^n satisfies the condition in (Step- n), completing the inductive step. \square

6.2. Pushing forward distributors. Let us recall some facts about higher associativity, which will be used later.

Definition 6.5 ([35, Definition 8.2]). Let M and N be topological monoids and $n \geq 1$ an integer. An A_n **structure** on a continuous map $f: M \rightarrow N$ is a family of maps $\delta_i: I^{i-1} \times M^i \rightarrow N$ for $1 \leq i \leq n$ satisfying the following:

- $\delta_1 = f$.
- For $2 \leq i \leq n$, the following boundary conditions hold:

$$\begin{cases} \delta_i(t_1, \dots, \overbrace{0}^{t_j}, \dots, t_{i-1}; x_1, \dots, x_i) = \delta_{i-1}(t_1, \dots, \widehat{t_j}, \dots, t_{i-1}; x_1, \dots, x_j x_{j+1}, \dots, x_i) \\ \delta_i(t_1, \dots, \overbrace{1}^{t_j}, \dots, t_{i-1}; x_1, \dots, x_i) = \delta_j(t_1, \dots, t_{j-1}; x_1, \dots, x_j) \delta_{i-j}(t_{j+1}, \dots, t_{i-1}; x_{j+1}, \dots, x_i). \end{cases}$$

Example 6.6. An A_1 structure on $f: M \rightarrow N$ consists of no additional data. An A_2 structure consists of paths

$$\delta^{x,y} := \delta_2(-; x, y): I \rightarrow N$$

from $f(xy)$ to $f(x)f(y)$, depending continuously on the inputs $x, y \in M$. In other words, the map f is A_2 if and only if it preserves the multiplication up to homotopy.

An A_3 structure on f consists of 1-cubes $\delta^{x,y}: I \rightarrow N$ and 2-cubes $\delta^{x,y,z}: I \rightarrow N$ as illustrated in Figure 6.1.

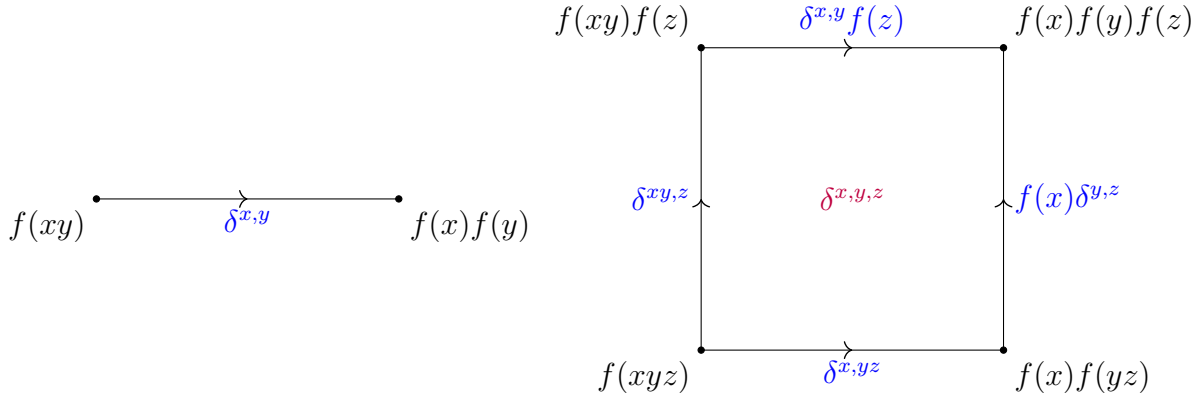


FIGURE 6.1. An A_3 structure on $f: M \rightarrow N$.

Remark 6.7. As in Remark 3.7, an A_n structure $\delta = (\delta_1, \dots, \delta_n)$ on a map $f: M \rightarrow N$ is determined by the highest dimensional part δ_n , since M is strictly unital. The same would not hold if M were merely unital up to homotopy.

Let us recast Definition 3.6 in this terminology. An n -distributor φ^n for a left linear \mathbf{Top}_* -enriched category \mathcal{T} consists of the following data: For each $a \in \mathcal{T}(A, B)$, an A_{n+1} structure φ_a for left multiplication by a , i.e., postcomposition

$$a_*: \mathcal{T}(X, A) \rightarrow \mathcal{T}(X, B),$$

with respect to addition in the mapping spaces $\mathcal{T}(X, A)$ and $\mathcal{T}(X, B)$. These A_{n+1} structures φ_a are required to depend continuously on $a \in \mathcal{T}$. Note that our indexing counts the

number of plus signs in $a(x_0 + \dots + x_n)$, which agrees with the dimension of the cube $\varphi_a^{x_0, \dots, x_n}: I^n \rightarrow \mathcal{T}(X, B)$.

The following result is due to Fuchs [15] and can be found in [35, §8]; c.f. [12, §4.3].

Lemma 6.8. (1) *A composition of A_n maps is an A_n map.*

(2) *A map homotopic to an A_n map is an A_n map.*

(3) *Let $f: M \xrightarrow{\cong} N$ be a homotopy equivalence between topological monoids. Then f is an A_n map if and only if any homotopy inverse $g: N \xrightarrow{\cong} M$ is A_n .*

Moreover, it follows from the explicit construction in [15] that the A_n structure of a composite gf depends continuously on the A_n structures of f and g .

Proposition 6.9 (Pushing forward distributors). *Let $F: \mathcal{S} \rightarrow \mathcal{T}$ be a morphism of left linear \mathbf{Top}_* -categories such that for all objects A, B of \mathcal{S} , the induced map $F: \mathcal{S}(A, B) \xrightarrow{\cong} \mathcal{T}(FA, FB)$ is a homotopy equivalence. Assume that the functor $\pi_0 F: \pi_0 \mathcal{S} \rightarrow \pi_0 \mathcal{T}$ is essentially surjective. If \mathcal{S} is N -distributive for some $N \geq 1$ (or $N = \infty$), then \mathcal{T} is N -distributive.*

Proof. Consider the factorization of $F: \mathcal{S} \rightarrow \mathcal{T}$ as

$$\mathcal{S} \rightarrow \mathrm{ObIm}(F) \rightarrow \mathcal{T},$$

where the “object-image” of the functor F is the full \mathbf{Top}_* -subcategory of \mathcal{T} consisting of objects of the form FX for some object X of \mathcal{S} . Then both functors $\mathcal{S} \rightarrow \mathrm{ObIm}(F)$ and $\mathrm{ObIm}(F) \rightarrow \mathcal{T}$ of this factorization satisfy the assumptions of the statement. This reduces the general statement to the following cases.

Case (a). F is surjective on objects.

Case (b). F is the identity on each mapping space, i.e., $F: \mathcal{S}(A, B) \xrightarrow{\cong} \mathcal{T}(FA, FB)$.

Proof for Case (a). For each object A of \mathcal{T} , choose an object denoted GA of \mathcal{S} satisfying $F GA = A$. For each pair of objects A, B of \mathcal{T} , choose a homotopy inverse $G: \mathcal{T}(A, B) \xrightarrow{\cong} \mathcal{S}(GA, GB)$ to the map $F: \mathcal{S}(GA, GB) \xrightarrow{\cong} \mathcal{T}(FA, FB)$, along with a homotopy $h: \mathcal{T}(A, B) \times I \rightarrow \mathcal{T}(A, B)$ from the identity to FG . Note that $G: \mathcal{T} \rightarrow \mathcal{S}$ is *not* a functor, as it preserves neither composition nor identities $1_A \in \mathcal{T}(A, A)$.

Since F preserves addition, it is in particular A_∞ with respect to addition. By Lemma 6.8(3), $G: \mathcal{T}(A, B) \rightarrow \mathcal{S}(GA, GB)$ admits an A_∞ -structure with respect to addition, which we denote γ . For any $x_0, \dots, x_n \in \mathcal{T}(X, A)$, we denote by $\gamma^{x_0, \dots, x_n}: I^n \rightarrow \mathcal{S}(GX, GA)$ the corresponding n -cube with extreme corners $G(x_0 + \dots + x_n)$ and $Gx_0 + \dots + Gx_n$.

Let φ^N be an N -distributor for \mathcal{S} . For every $a \in \mathcal{T}(A, B)$, consider $Ga \in \mathcal{S}(GA, GB)$ and the composite

$$\begin{aligned} \mathcal{T}(X, A) &\xrightarrow{G} \mathcal{S}(GX, GA) \xrightarrow{(Ga)_*} \mathcal{S}(GX, GB) \\ x &\longmapsto Gx \longmapsto (Ga)(Gx) \end{aligned}$$

of G and left multiplication by Ga , both of which are A_{n+1} with respect to addition. By Lemma 6.8(1), this composite inherits an A_{n+1} -structure with respect to addition, which we denote ξ^n . This A_{n+1} structure ξ^n depends continuously on the element $Ga \in \mathcal{S}(GA, GB)$, and therefore on $a \in \mathcal{T}(A, B)$.

For instance, $\xi_a^{x,y}: I^1 \rightarrow \mathcal{S}(GX, GB)$ is the concatenation of paths:

$$(Ga)G(x+y) \xrightarrow{(Ga)\gamma^{x,y}} (Ga)(Gx+Gy) \xrightarrow{\varphi_{Ga}^{Gx,Gy}} GaGx + GaGy.$$

Now we prove the statement by induction, using the following condition for $n \leq N$.

(Step- n)

- There is given an n -distributor ψ^n for \mathcal{T} , based on ψ^{n-1} .
- There is given a homotopy

$$h^n: \psi^n \simeq F(\xi^n)$$

which is compatible with the previous steps in the following sense. For every proper subcube $C_\sigma \subset I^n$, of dimension $\dim \sigma = d < n$, the restriction of h^n to C_σ satisfies

$$h^n|_{C_\sigma \times I} = h^d[\sigma]: (\psi^d)[\sigma] \simeq F(\xi^d[\sigma]).$$

Base case $n = 0$. The 0-distributor ψ^0 for \mathcal{T} is forced to be $\psi_a^x = ax$. For the homotopy $h^0: \psi^0 \simeq F(\xi^0)$, take the path

$$ax \xrightarrow{(h^0)_a^x := h_a h_x} (FGa)(FGx).$$

where $h_a := h(a, -): I \rightarrow \mathcal{T}(A, B)$ is the path following a throughout the homotopy $h: \mathcal{T}(A, B) \times I \rightarrow \mathcal{T}(A, B)$. The compatibility condition on h^0 is vacuous, since I^0 has no proper subcube.

Inductive step from $n - 1$ to n . By induction hypothesis and Lemma 6.2, the given homotopies h^{n-1} define a homotopy

$$\mathcal{O}(h^{n-1}): \mathcal{O}(\psi^{n-1}) \simeq F\mathcal{O}(\xi^{n-1}).$$

By Lemma 4.7, the map $\partial I^n \times \mathcal{T}(A, B) \times \mathcal{T}(X, A)^{n+1} \hookrightarrow I^n \times \mathcal{T}(A, B) \times \mathcal{T}(X, A)^{n+1}$ is a Hurewicz cofibration. By the homotopy extension property, there is a homotopy

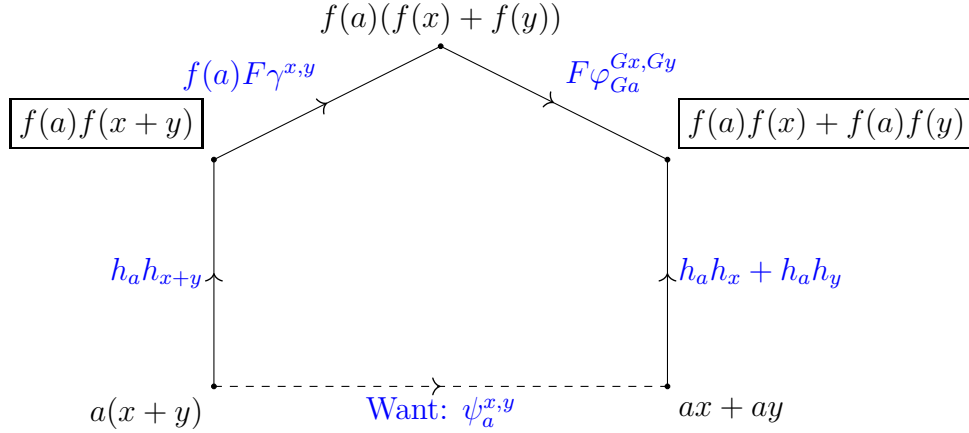
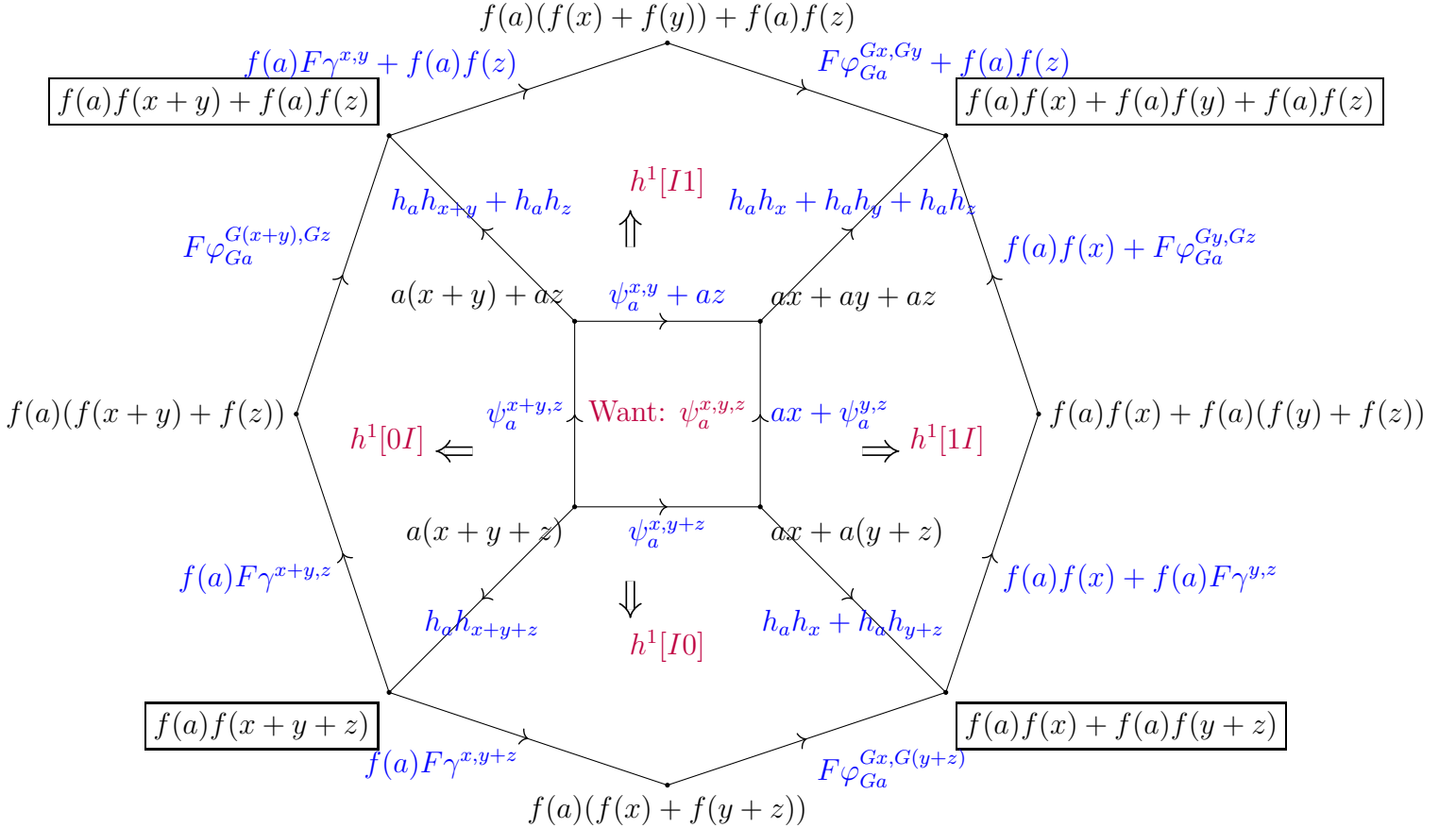
$$\tilde{h}^n: I^n \times \mathcal{T}(A, B) \times \mathcal{T}(X, A)^{n+1} \times I \rightarrow \mathcal{T}(X, B)$$

extending $\mathcal{O}(h^{n-1})$ and ending at $F(\xi^n)$. Denote the start of the homotopy by $\psi^n := \tilde{h}_0^n$, which satisfies

$$\psi^n|_{\partial I^n} = \tilde{h}_0^n|_{\partial I^n} = \mathcal{O}(h^{n-1})_0 = \mathcal{O}(\psi^{n-1}),$$

so that ψ^n is an n -distributor for \mathcal{T} based on ψ^{n-1} . Moreover, the homotopy $\tilde{h}^n: \psi^n \simeq F(\xi^n)$ satisfies the compatibility condition required in the induction hypothesis.

The inductive steps to $n = 1$ and $n = 2$ are illustrated in Figures 6.2 and 6.3. To ease the notation there, we denote $f(x) := FGx \in \mathcal{T}$, with the paths $h_x: x \rightarrow f(x)$.

FIGURE 6.2. The homotopy $\mathcal{O}(h^0)$ from $\mathcal{O}(\psi^0)$ to $F(\xi^0): \partial I^1 \rightarrow \mathcal{T}$.FIGURE 6.3. The homotopy $\mathcal{O}(h^1)$ from $\mathcal{O}(\psi^1)$ to $F(\xi^1): \partial I^2 \rightarrow \mathcal{T}$.

Proof for Case (b). The functor $\pi_0 F: \pi_0 \mathcal{S} \rightarrow \pi_0 \mathcal{T}$ is an equivalence of categories. Choose and inverse equivalence $G: \pi_0 \mathcal{T} \rightarrow \pi_0 \mathcal{S}$, with a natural isomorphism $\epsilon: (\pi_0 F)G \xrightarrow{\cong} \text{id}_{\pi_0 \mathcal{T}}$. For

every object X of \mathcal{T} , consider the inverse isomorphisms

$$\begin{cases} \epsilon_X \in (\pi_0 \mathcal{T})(FGX, X) \\ \epsilon_X^{-1} \in (\pi_0 \mathcal{T})(X, FGX). \end{cases}$$

and choose representative maps

$$\begin{cases} \widetilde{\epsilon}_X \in \mathcal{T}(FGX, X) \\ \widetilde{\epsilon}_X^{-1} \in \mathcal{T}(X, FGX). \end{cases}$$

By construction, these maps $\widetilde{\epsilon}_X: FGX \xrightarrow{\sim} X$ and $\widetilde{\epsilon}_X^{-1}: X \xrightarrow{\sim} FGX$ are inverse homotopy equivalences, and hence induce homotopy equivalences on mapping spaces, upon applying functors of the form $\mathcal{T}(W, -)$ or $\mathcal{T}(-, Z)$. Now let X, A, B be objects of \mathcal{T} and consider the diagram:

$$\begin{array}{ccc} \partial I^n \times \mathcal{S}(GA, GB) \times \mathcal{S}(GX, GA)^{n+1} & & \mathcal{S}(GX, GB) \\ \parallel & & \parallel \\ \partial I^n \times \mathcal{T}(FGA, FGB) \times \mathcal{T}(FGX, FGA)^{n+1} & \xrightarrow{\mathcal{O}(\varphi^{n-1})} & \mathcal{T}(FGX, FGB) \\ \downarrow \text{id} \times (\widetilde{\epsilon}_B)_* (\widetilde{\epsilon}_A^{-1})^* \times (\widetilde{\epsilon}_A)_* (\widetilde{\epsilon}_X^{-1})^* \simeq & \searrow \varphi^n & \downarrow \simeq (\widetilde{\epsilon}_B)_* (\widetilde{\epsilon}_X^{-1})^* \\ I^n \times \mathcal{T}(FGA, FGB) \times \mathcal{T}(FGX, FGA)^{n+1} & & \mathcal{T}(X, B) \\ \downarrow \mathcal{O}(\psi^{n-1}) & \searrow \psi^n & \\ \partial I^n \times \mathcal{T}(A, B) \times \mathcal{T}(X, A)^{n+1} & \xrightarrow{\mathcal{O}(\psi^{n-1})} & \mathcal{T}(X, B) \\ \downarrow & & \\ I^n \times \mathcal{T}(A, B) \times \mathcal{T}(X, A)^{n+1} & & \end{array}$$

where the back and front right faces need not commute strictly. By the same inductive argument as in Case (a), we can push forward the n -distributor φ^n for \mathcal{S} along the downward homotopy equivalences to produce an n -distributor ψ^n for \mathcal{T} . \square

Corollary 6.10 (Homotopy invariance). *Let $F: \mathcal{S} \rightarrow \mathcal{T}$ be a morphism of left linear \mathbf{Top}_* -categories which is moreover a Dwyer–Kan equivalence, i.e., for all objects A, B of \mathcal{S} , the map $F: \mathcal{S}(A, B) \xrightarrow{\sim} \mathcal{T}(FA, FB)$ is a weak equivalence, and the functor $\pi_0 F: \pi_0 \mathcal{S} \xrightarrow{\sim} \pi_0 \mathcal{T}$ is an equivalence of categories.*

Assume that all mapping spaces in \mathcal{S} and in \mathcal{T} have the homotopy type of a CW complex. Then for every $n \geq 1$ (or $n = \infty$), \mathcal{S} is n -distributive if and only if \mathcal{T} is n -distributive.

Proof. Recall that a weak equivalence between spaces of the homotopy type of a CW complex is automatically a homotopy equivalence. Now the result follows from Propositions 6.4 and 6.9. \square

APPENDIX A. MODELS FOR SPECTRA

In this section, we recall the construction of *Bousfield–Friedlander spectra* as described in [13, §2.1], along with some of their properties.

Notation A.1. Let $s\mathbf{Set}_*$ denote the category of pointed simplicial sets. Let $\Sigma: s\mathbf{Set}_* \rightarrow s\mathbf{Set}_*$ denote the reduced suspension functor, given by the smash product $\Sigma X = S^1 \wedge X$, using the model of the circle $S^1 = \Delta^1/\partial\Delta^1$. Note that this suspension is *not* the Kan suspension [16, §III.5].

Definition A.2. (1) A **spectrum** $X = \{X_n\}_{n \geq 0}$ is a sequence of pointed simplicial sets X_n together with structure maps $\sigma_n: \Sigma X_n \rightarrow X_{n+1}$, or equivalently by adjunction, maps $\tilde{\sigma}_n: X_n \rightarrow \Omega X_{n+1}$. We call X_n the simplicial set in **level** n .
 (2) A **map** of spectra $f: X \rightarrow Y$ consists of a sequence of maps $f_n: X_n \rightarrow Y_n$ for $n \geq 0$ such that the diagrams

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \sigma_n^X \downarrow & & \downarrow \sigma_n^Y \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

commute (strictly).

- (3) An **Ω -spectrum** is a spectrum X whose structure maps $\tilde{\sigma}_n: X_n \xrightarrow{\sim} \Omega X_{n+1}$ are weak equivalences for all $n \geq 0$.
 (4) For $i \in \mathbb{Z}$, the i^{th} **homotopy group** of a spectrum X is

$$\pi_i X := \operatorname{colim}_n \pi_{i+n} X_n$$

where colimit is of the sequence of groups $(\tilde{\sigma}_n)_*: \pi_{i+n} X_n \rightarrow \pi_{i+n} \Omega X_{n+1} = \pi_{i+n+1} X_{n+1}$. Let **Spec** denote the category of spectra.

Definition A.3. The *stable model structure* on **Spec** is defined as follows.

- The **stable equivalences** are the π_* -isomorphisms.
- The **stable cofibrations** are the strict cofibrations, i.e., maps $f: X \rightarrow Y$ such that $f_0: X_0 \rightarrow Y_0$ and

$$X_{n+1} \bigcup_{\Sigma X_n} \Sigma Y_n \rightarrow Y_{n+1}$$

are cofibrations in $s\mathbf{Set}_*$ for all $n \geq 0$. In particular, an object X is cofibrant if and only if its structure maps $\Sigma X_n \rightarrow X_{n+1}$ are cofibrations in $s\mathbf{Set}$.

- *Stable fibrations* are such that an object X is fibrant if and only if X is an Ω -spectrum and levelwise fibrant.

We collect some salient facts about the category **Spec**.

Theorem A.4. [13, Theorem 2.3] *The stable model structure makes **Spec** into a simplicial model category.*

In particular, **Spec** is enriched in (pointed) simplicial sets, where the function complex **Spec** (X, Y) has n -simplices

$$\underline{\mathbf{Spec}}(X, Y)_n \cong \operatorname{Hom}_{\mathbf{Spec}}(X \otimes \Delta^n, Y)$$

as described in [16, §II.2]. Via geometric realization, this yields a **Top** $_*$ -enriched category **Spec**, with mapping spaces

$$\mathbf{Spec}(X, Y) := |\underline{\mathbf{Spec}}(X, Y)|.$$

Remark A.5. **Spec** is Quillen equivalent to the other model categories of spectra, for instance Kan’s prespectra, Kan’s semisimplicial spectra [13, §2.5], symmetric spectra, and orthogonal spectra [25, Theorem 0.1]. Note that by Schwede’s rigidity theorem, any two model categories whose homotopy categories are triangulated-equivalent to the stable homotopy category are Quillen equivalent [32].

Lemma A.6. *For any simplicial sets S and T , the natural map $\Sigma(S \times T) \rightarrow \Sigma S \times \Sigma T$ is a monomorphism (i.e., a cofibration).*

Proof. For every $k \geq 0$, the k -simplices of S^1 are given by

$$(S^1)_k = \Delta([k], [1])/c_0 \sim c_1$$

where c_i denotes the constant function with value i . The k -simplices of the suspension ΣT are

$$(\Sigma T)_k = \bigvee_{i=1}^k T_k.$$

The map $\theta: \Sigma(S \times T) \rightarrow \Sigma S \times \Sigma T$ has in simplicial degree k the map of pointed sets

$$\begin{array}{ccc} (\Sigma(S \times T))_k & \xrightarrow{\theta_k} & (\Sigma S \times \Sigma T)_k \\ \parallel & & \parallel \\ \bigvee_{i=1}^k (S_k \times T_k) & & \left(\bigvee_{j=1}^k S_k \right) \times \left(\bigvee_{j'=1}^k T_k \right) \end{array}$$

whose restriction to the i^{th} summand is the product of summand inclusions

$$\text{inc}_i \times \text{inc}_i: S_k \times T_k \rightarrow \bigvee_{j=1}^k S_k \times \bigvee_{j'=1}^k T_k,$$

so that θ_k is injective. □

Lemma A.7. *If X and Y are cofibrant spectra, then the natural map $\iota: X \vee Y \rightarrow X \times Y$ is a cofibration. In particular, $X \times Y$ is cofibrant.*

Proof. In level 0, we have

$$\iota_0: X_0 \vee Y_0 \rightarrow X_0 \times Y_0$$

which is a cofibration in $s\mathbf{Set}_*$, i.e., a monomorphism. Given $n \geq 0$, consider the commutative diagram in $s\mathbf{Set}_*$

$$\begin{array}{ccc} \Sigma(X \vee Y)_n & \xrightarrow{\Sigma \iota_n} & \Sigma(X \times Y)_n \\ \parallel & & \downarrow \\ \Sigma X_n \vee \Sigma Y_n & \xrightarrow{\quad} & \Sigma X_n \times \Sigma Y_n \\ \sigma_n^X \vee \sigma_n^Y \downarrow & & \downarrow \sigma_n^X \times \sigma_n^Y \\ X_{n+1} \vee Y_{n+1} = (X \vee Y)_{n+1} & \xrightarrow{\quad} & (X \times Y)_{n+1} = X_{n+1} \times Y_{n+1}. \end{array}$$

We want to show that the map of simplicial sets

$$X_{n+1} \vee Y_{n+1} \bigcup_{\Sigma X_n \vee \Sigma Y_n} \Sigma(X_n \times Y_n) \rightarrow X_{n+1} \times Y_{n+1}$$

is a cofibration. This map is a composite

$$X_{n+1} \vee Y_{n+1} \bigcup_{\Sigma X_n \vee \Sigma Y_n} \Sigma(X_n \times Y_n) \longrightarrow X_{n+1} \vee Y_{n+1} \bigcup_{\Sigma X_n \vee \Sigma Y_n} \Sigma X_n \times \Sigma Y_n \longrightarrow X_{n+1} \times Y_{n+1}$$

where the first step is a cofibration, since $\Sigma(X_n \times Y_n) \hookrightarrow \Sigma X_n \times \Sigma Y_n$ is a cofibration, by Lemma A.6. One readily checks that the second step is a monomorphism (hence a cofibration), using the fact that limits and colimits of simplicial sets are computed degreewise.

Finally, the map $* \rightarrow X \times Y$ is a composite of cofibrations

$$* \hookrightarrow X \hookrightarrow X \vee Y \hookrightarrow X \times Y$$

and thus $X \times Y$ is cofibrant. \square

Lemma A.8. *Let A be an abelian group. Then there is an Eilenberg–MacLane spectrum HA in **Spec** which is fibrant, cofibrant, and an abelian group object in **Spec**, with addition map $+: HA \times HA \rightarrow HA$ compatible with the addition map of A .*

*Moreover, if A is an \mathbb{F}_p vector space, then HA is an \mathbb{F}_p -vector space object in **Spec**.*

Proof. See Appendix B. \square

Let $\text{sh}: \mathbf{Spec} \rightarrow \mathbf{Spec}$ denote the **shift functor** of spectra, defined by $\text{sh}(X)_n = X_{n+1}$. The shift has the homotopy type of the suspension $\text{sh}X \simeq \Sigma X$. In particular, we obtain other Eilenberg–MacLane spectra $K_n^A := \text{sh}^n HA \simeq \Sigma^n HA$ which are also Ω -spectra (hence fibrant), cofibrant, and abelian group objects. Moreover, a finite product of objects $K_{n_i}^{A_i}$ is also a fibrant cofibrant abelian group object in **Spec**. Using this, we obtain:

Proof of Proposition 2.7. Let \mathcal{T} be the full subcategory of **Spec** consisting of finite products

$$K = \prod_i K_{n_i}^{A_i}.$$

By construction, \mathcal{T} has finite products, which are the same as in the ambient category **Spec**. Since each object K of \mathcal{T} is an abelian group object in **Spec**, the mapping space $\mathbf{Spec}(X, K)$ is a topological abelian group, with pointwise addition in the target K . Hence, \mathcal{T} is a left linear mapping theory, and F_X is a left linear model of it.

For objects K and L in \mathcal{T} , the natural map $\iota: K \vee L \rightarrow K \times L$ is a cofibration in **Spec**. Since **Spec** is a simplicial model category, the restriction map

$$\begin{array}{ccc} \underline{\mathbf{Spec}}(K \times L, Z) & \xrightarrow{\iota^*} & \underline{\mathbf{Spec}}(K \vee L, Z) \\ & \searrow (i_K^*, i_L^*) & \downarrow \cong \\ & & \underline{\mathbf{Spec}}(K, Z) \times \underline{\mathbf{Spec}}(L, Z) \end{array}$$

is a Kan fibration for any fibrant object Z in **Spec**, in particular for any object in \mathcal{T} . Moreover, the map $\iota: K \vee L \xrightarrow{\sim} K \times L$ is a weak equivalence. Hence, the restriction map ι^* is a weak equivalence [16, Lemma II.4.2]. To conclude, use the fact that the geometric realization of a Kan fibration is a Serre fibration [16, Theorem 10.10]. \square

Remark A.9. We could have worked with other models of spectra. As discussed in [13, §2.5], there is an analogously defined category $\mathbf{Spec}_{\mathbf{Top}}$ of *topological spectra*, where $s\mathbf{Set}$ is replaced by **Top**. The Quillen equivalence $|-|: s\mathbf{Set} \rightleftarrows \mathbf{Top}: \text{Sing}$ induces a Quillen equivalence

$$|-|: \mathbf{Spec} \rightleftarrows \mathbf{Spec}_{\mathbf{Top}}: \text{Sing}.$$

Given a fibrant cofibrant abelian group object HA in \mathbf{Spec} , its geometric realization $|HA|$ is a fibrant cofibrant abelian group object in $\mathbf{Spec}_{\mathbf{Top}}$.

There is also the category $\mathcal{N}\mathbf{Spec}$ of \mathcal{N} -spectra, described in [25, Definition 1.9, Theorem 2.2, Example 4.1]. The category $\mathcal{N}\mathbf{Spec}$ is isomorphic to $\mathbf{Spec}_{\mathbf{Top}}$. The *stable model structure* described in [25, Definition 9.1, Theorem 9.2] coincides with the stable model structure of Bousfield–Friedlander on $\mathbf{Spec}_{\mathbf{Top}}$. The difference is that $\mathcal{N}\mathbf{Spec}$ is constructed as a *topological* model category, without going through simplicial sets. Mapping spaces in the two categories are related as follows:

$$\underline{\mathbf{Spec}}_{\mathbf{Top}}(X, Y) = \mathrm{Sing} \mathcal{N}\mathbf{Spec}(X, Y).$$

Remark A.10. It was pointed out to us by Irakli Patchkoria and Stefan Schwede that a model for the Eilenberg–MacLane spectrum HA as in Lemma A.8 can also be obtained in symmetric spectra of simplicial sets, endowed with the absolute flat stable model structure [33], also called the S model structure in [19, Definition 5.3.6].

For the record, we extract a more general statement from the proof of Proposition 2.7.

Example A.11. Let \mathcal{C} be a pointed simplicial model category [31, §II.2] [16, §II.3]. View \mathcal{C} as being \mathbf{Top}_* -enriched by taking the geometric realization of the simplicial mapping spaces:

$$\mathcal{C}(A, B) := |\underline{\mathcal{C}}(A, B)|.$$

Assume that \mathcal{C} satisfies the following: For any cofibrant objects X and Y , the map $\iota: X \vee Y \rightarrow X \times Y$ is a cofibration; in particular, $X \times Y$ is cofibrant.

Let $\mathcal{G} \subseteq \mathrm{Ob} \mathcal{C}$ be a set of abelian group objects in \mathcal{C} which are fibrant and cofibrant. Assume moreover that for any A, B in \mathcal{G} , the map $A \vee B \rightarrow A \times B$ is a weak equivalence. Then the full subcategory $\mathcal{T}_{\mathcal{G}}$ of \mathcal{C} consisting of finite products of objects of \mathcal{G} is a weakly bilinear mapping theory.

Note that every object in $\mathcal{T}_{\mathcal{G}}$ is fibrant and cofibrant as an object in \mathcal{C} , so that the mapping spaces $\mathcal{C}(A, B)$ are derived mapping spaces.

For any object X of \mathcal{C} , we obtain the left linear model $F_X = \mathcal{C}(X, -): \mathcal{T}_{\mathcal{G}} \rightarrow \mathbf{Top}_*$. If moreover X is cofibrant, then $\mathcal{C}(X, A)$ is a derived mapping space.

APPENDIX B. CONSTRUCTION OF EILENBERG–MACLANE SPECTRA

There are different constructions of the Eilenberg–MacLane spectrum HA as an Ω -spectrum, whose constituent spaces are Eilenberg–MacLane spaces $(HA)_n = K(A, n)$. The specific features of HA depend on the specific construction of Eilenberg–MacLane spaces.

Iterated classifying space. The following argument was kindly provided to us by Marc Stephan. More details can be found in [36, Part I].

Recall the following construction of the classifying space of a simplicial group (which is the same as a group object in simplicial sets). Consider the functor $B: s\mathbf{Gp} \rightarrow s\mathbf{Set}$ given by

$$BG := \mathrm{diag} B_{\bullet}(*, G, *)$$

where $B_{\bullet}(X, G, Y)$ is the two-sided bar construction [27, §7], which is a bisimplicial set, and diag denotes its diagonal. Explicitly, $B(*, G, *)$ has in external degree n the simplicial set $B_n(*, G, *) = G^n$, so that BG has as k -simplices the set $(BG)_k = (G_k)^k$. Note that B preserves finite products and thus induces a functor on abelian group objects $B: s\mathbf{Ab} \rightarrow s\mathbf{Ab}$.

Proof of Lemma A.8. Starting from an abelian group A , viewed as a constant simplicial abelian group, iterating the functor B yields Eilenberg–MacLane spaces $B^n A \simeq K(A, n)$. Form a spectrum HA defined by $(HA)_n := B^n A$. The structure maps $\sigma_n: S^1 \wedge B^n A \rightarrow B^{n+1} A$ have in simplicial degree k the inclusion

$$\bigvee_{i=1}^k (B^n A)_k \rightarrow \prod_{i=1}^k (B^n A)_k.$$

In particular, $S^1 \wedge B^n A \rightarrow B^{n+1} A$ is a cofibration of simplicial sets for each $n \geq 0$, so that HA is cofibrant. Moreover, HA is an Ω -spectrum and each simplicial set $HA_n = B^n A$ is a Kan complex, since it is a simplicial group. Therefore, HA is fibrant. Also, $B^n A$ is an abelian group object in $s\mathbf{Set}$ for each $n \geq 0$, and the structure maps $\sigma_n: S^1 \wedge B^n A \rightarrow B^{n+1} A$ are linear in the factor $B^n A$, so that the adjunct structure maps

$$\tilde{\sigma}_n: B^n A \xrightarrow{\sim} \Omega B^{n+1} A$$

are maps of simplicial abelian groups.

Moreover, if A is an \mathbb{F}_p -vector space, then each simplicial abelian group $B^n A$ is a simplicial \mathbb{F}_p -vector space, and thus HA is an \mathbb{F}_p -vector space object. \square

Remark B.1. The construction above has an analogue in topological spaces. Consider the classifying space functor

$$B: \mathbf{TopGp} \rightarrow \mathbf{Top}_*$$

given by geometric realization of a simplicial space $B(G) = |B_\bullet(*, G, *)|$; see [28, §16.5] for more details. If G is a simplicial group, then the two constructions of the classifying space are related by $|BG| = B|G|$.

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